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The Riemann-Hilbert Approach to Initial-Boundary Value Problems for Integrable Coherently Coupled Nonlinear Schrödinger Systems on the Half-Line

Beibei Hu[1,2], Tiecheng Xia[1,*] and Wen-Xiu Ma[3,4,5]

[1] *Department of Mathematics, Shanghai University, Shanghai, 200444, China.*
[2] *School of Mathematics and Finance, Chuzhou University, Anhui, 239000, China.*
[3] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL, 33620-5700, USA.*
[4] *Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho, 2735, South Africa.*
[5] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong, 266590, China.*

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Abstract. An integrable coherently coupled nonlinear Schrödinger system describing the propagation of polarised optical waves in an isotropic medium with a generalized 4×4 matrix Ablowitz-Kaup-Newell-Segur-type Lax pair is studied. The corresponding initial-boundary value problem is reduced to a matrix Riemann-Hilbert problem in the complex plane. Moreover, it is shown that the associated spectral functions depend on each other and satisfy a global relationship.

AMS subject classifications: 35G31, 35Q51, 35Q15

Key words: Riemann-Hilbert problem, coherently coupled nonlinear Schrödinger system, initial-boundary value problem, unified transform method.

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$iq_t \pm q_{xx} + |q|^2 q = 0,$$

arises in plasma physics, solid-state physics, nonlinear optics and water waves. It describes the propagation of optical solitons in mono-mode fibers for scalar fields and the dependance of such solitons on the group velocity dispersion (GVD) and the self-phase modulation (SPM) [11]. Since nonlinear phase change comes from the cross-phase modulation

*Corresponding author. *Email addresses:* hu_chzu@163.com, hubsquare@chzu.edu.cn (B. Hu), xiatc@shu.edu.cn (T. Xia), mawx@cas.usf.edu (W. X. Ma)

(XPM) in birefringent or multi-mode fibers, the interaction of several field components at different frequencies or polarisations has to be taken into account. The dynamic features of such solitons are usually governed by coupled nonlinear Schrödinger (CNLS) systems [11]. Multicomponent solitons (MSs) are intriguing nonlinear objects where soliton is split into a number of components. These solitons are called the vector or the multicolour solitons.

Let us note that in optical fibers there are two types of vector solitons — viz. coherently and incoherently coupled vector solitons [11]. For incoherently coupled vector solitons the coupling is phase insensitive and special incoherently CNLS system —- the Manakov system, has the following form

$$\begin{aligned} &iu_\zeta \pm \frac{1}{2}u_{\tau\tau} + (|u|^2+|v|^2)u = 0, \\ &iv_\zeta \pm \frac{1}{2}v_{\tau\tau} + (|u|^2+|v|^2)v = 0, \end{aligned} \tag{1.1}$$

where ζ and τ, respectively, refer to the normalised spatial and temporal coordinates, the sign $+$ or $-$ corresponds to the anomalous dispersion (bright soliton) or normal dispersion (dark soliton) regime. Besides, $|u|^2u$ and $|v|^2v$ denote SPM effects, whereas the XPM effects $|u|^2v$ and $|v|^2u$ serve as incoherent coupling terms [15].

The initial-boundary value (IBV) problems for the system (1.1) on the half-line have been recently studied by using the Fokas method [10, 36]. This method can be also employed to consider the IBV problems for linear and nonlinear integrable evolution PDEs with 2×2 Lax pairs [6–8, 20, 21, 35, 42]. Similar to IST on a line, the Fokas approach allows to express the solutions of IBV problems via solutions of Riemann-Hilbert (RH) problems. Lenells [22] extended the Fokas approach to IBV problems for integrable nonlinear evolution equations with 3×3 Lax pairs. This stimulated the study of IBV problems with the Lax pairs of higher-order such as, the Degasperis-Procesi equation [23], the Ostrovsky-Vakhnenko equation [24], the Sasa-Satsuma equation [37], the three wave equation [38], the spin-1 Gross-Pitaevskii equation [40] and others [19,25,30]. Integrable equations with 2×2 or 3×3 Lax pairs have been also studied [12–14,42]. In particular, Deift and Zhou [5] investigated the asymptotic of the solutions by applying the steepest descent method to a RH problem.

There are also vector solitons associated with coherent CNLS systems. They can be used as the carriers of the switched information in optical fields [11]. The coupling effects depend on relative phases of the interacting fields, and coherent interactions usually occur when the nonlinear medium is weakly anisotropic or low birefringent [11]. Park and Shin [28] proposed new integrable CNLS equations — viz.

$$\begin{aligned} &iu_t + u_{xx} + 2(|u|^2+2|v|^2)u - 2u^*v^2 = 0, \\ &iv_t + v_{xx} + 2(2|u|^2+|v|^2)v - 2v^*u^2 = 0, \end{aligned} \tag{1.2}$$

where u and v denote slowly varying envelopes of two interacting optical modes, x and t are, respectively, the normalised distance and time, and $*$ means the complex conjugation. Zhang *et al.* [41] used the Ablowitz-Kaup-Newell-Segur (AKNS) technology [1] to establish