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A New Error Analysis of Nonconforming EQ_1^{rot} FEM for Nonlinear BBM Equation

Yanhua Shi[1], Yanmin Zhao[1,*], Fenling Wang[1] and Dongyang Shi[2]

[1] *School of Mathematics and Statistics, Xuchang University, Xuchang 461000, Henan, China*
[2] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, Henan, China*

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Abstract. Nonconforming EQ_1^{rot} element is applied to solving a kind of nonlinear Benjamin-Bona-Mahony (BBM for short) equation both for space-discrete and fully discrete schemes. A new important estimate is proved, which improves the result of previous works with the exact solution u belonging to $H^2(\Omega)$ instead of $H^3(\Omega)$. And then, the supercloseness and global superconvergence estimates in broken H^1 norm are obtained for space-discrete scheme. Further, the superclose estimates are deduced for backward Euler and Crank-Nicolson schemes. To confirm our theoretical analysis, numerical experiments for backward Euler scheme are executed. It seems that the results presented herein have never been seen for nonconforming FEMs in the existing literature.

AMS subject classifications: 65N15, 65N30

Key words: BBM equation, nonconforming FEM, space-discrete and fully discrete schemes, supercloseness and superconvergence.

1 Introduction

In this paper, we mainly pay attention to the following BBM equation:

$$\begin{cases} u_t-\Delta u_t=\nabla\cdot\vec{f}(u), & (X,t)\in\Omega\times(0,T], \\ u(X,t)=0, & (X,t)\in\partial\Omega\times(0,T], \\ u(X,0)=u_0(X), & X\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^2$ is an open bounded convex polygonal domain with boundary $\partial\Omega$, $X=(x,y)$, $\vec{f}(u)=-(\frac{1}{2}u^2+u,\frac{1}{2}u^2+u)$, $\nabla\cdot$ denotes the divergence operator.

*Corresponding author.
Email: zhaoym@lsec.cc.ac.cn (Y. M. Zhao)

In fluid mechanics, many interesting equations of mathematical physics have been presented to describe the unidirectional propagation of long waves. For researching water waves, BBM equation $u_t - u_{xxt} + u_x + uu_x = 0$ was introduced in [1] as a well-known model for the surface wave in a channel and consequently a lot of work have been devoted to this equation. For example, [2] and [3] discussed the periodic solutions and solitary wave solutions of a kind of BBM problem, respectively. [4] performed Lie symmetry classification of a time-coefficient BBM equation and some exact solutions were derived. As to numerical methods, [5] proposed a new multi-symplectic Euler box scheme and implemented the error analysis. [6] studied a Crank-Nicolson-type finite difference scheme and derived optimal error estimates with the second order. In multi-dimensional case, [7] proved the global existence and uniqueness of the solution when space dimension n equals from 1 to 5. [8] and [9] extended the results of [7] to all dimensions. [10] researched a Crank-Nicolson conforming FEM for (1.1), when n equals $1,2,3$ and proved the same optimal order estimates as [6]. [11] investigated a fully-discrete Galerkin scheme for DGRLW equation based on nonlinear Crank-Nicolson scheme, then a linearized modification scheme by an extrapolation method was discussed and the error estimates were given. But up to now, we find that there are limited studies about nonconforming FEM for Problem (1.1).

As we know, EQ_1^{rot} FE is an important nonconforming rectangular FE introduced by [12,13] with two special properties. One is that the consistency error can be estimated by $\mathcal{O}(h^2)$ order when the exact solution belongs to $H^3(\Omega)$, which is one order higher than its interpolation error (h is the space step). The other is that the interpolation operator is equivalent to projection operator. The element has been applied to solving many problems [14–22]. In more details, [14] studied optimal order estimate, supercloseness and the global superconvergence for linear viscoelasticity euqation. [15] discussed planar linear elasticity problem and proved an optimal order estimate. [16] and [17] applied this element to solving linear diffusion-convection-reaction equation and analyzed the convergence and superconvergence, respectively. For nonlinear case, [18,19] researched Sobolev equation and obtained the same results as [14]. [20] concentrated on a generalized elliptic problem on anisotropic meshes and deduced some new high-accuracy estimates. [21] and [22] investigated the elliptic problem and Signorini problem on quadrilateral meshes, respectively.

In the FEM analysis of some nonlinear equations, a posterior error hypothesis is required usually, that is, $\|u-u_h\|_{0,\infty} \leq 1$, where u_h is approximation of exact solution u. For example, [23] concentrated on the approximation to Cahn-Hilliard equation with cubic Hermite FE, and an optimal error estimate in L^2-norm was obtained for a linearized backward Euler scheme. [24] researched a mixed FEM for Schrödinger problem by the pair of bilinear and Nédélećs elements, and presented the error analysis of related variables for backward Euler and Crank-Nicolson schemes, respectively.

In this paper, we will try to apply EQ_1^{rot} FE to problem (1.1) for space-discrete and fully-discrete schemes. Firstly, an important estimate with order $\mathcal{O}(h^2)$ is proved when the solution u belongs to $H^2(\Omega)$ instead of $H^3(\Omega)$ and the partition mesh does not need to