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Generalized Inverse Analysis on the Domain $\Omega(A,A^+)$ in $B(E,F)$

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Abstract. Let $B(E,F)$ be the set of all bounded linear operators from a Banach space E into another Banach space $F,B^+(E,F)$ the set of all double splitting operators in $B(E,F)$ and $GI(A)$ the set of generalized inverses of $A\in B^+(E,F)$. In this paper we introduce an unbounded domain $\Omega(A,A^+)$ in $B(E,F)$ for $A\in B^+(E,F)$ and $A^+\in GI(A)$, and provide a necessary and sufficient condition for $T\in\Omega(A,A^+)$. Then several conditions equivalent to the following property are proved: $B=A^+(I_F+(T-A)A^+)^{-1}$ is the generalized inverse of T with $R(B)=R(A^+)$ and $N(B)=N(A^+)$, for $T\in\Omega(A,A^+)$, where I_F is the identity on F. Also we obtain the smooth (C^∞) diffeomorphism $M_A(A^+,T)$ from $\Omega(A,A^+)$ onto itself with the fixed point A. Let $S=\{T\in\Omega(A,A^+):R(T)\cap N(A^+)=\{0\}\}$, $M(X)=\{T\in B(E,F):TN(X)\subset R(X)\}$ for $X\in B(E,\mathcal{F})\}$, and $\mathcal{F}=\{M(X):\forall X\in B(E,F)\}$. Using the diffeomorphism $M_A(A^+,T)$ we prove the following theorem: S is a smooth submanifold in $B(E,F)$ and tangent to $M(X)$ at any $X\in S$. The theorem expands the smooth integrability of $\mathcal{F}$ at A from a local neighborhoold at A to the global unbounded domain $\Omega(A,A^+)$. It seems to be useful for developing global analysis and geomatrical method in differential equations.

Key Words: Generalized inverse analysis, smooth diffeomorphism, smooth submanifold.

AMS Subject Classifications: 47B38, 15A29, 58A05

1 Introduction

Let E, F be two Banach spaces, $B(E,F)$ the set of all linear bounded operators from E into $F,B^+(E,F)$ that of all double splitting operators in $B(E,F)$, and $GI(A)$ that of all generalized inverses of A for $A\in B^+(E,F)$. Write $V(A,A^+)=\{T\in B(E,F):\|T-A\|<\|A^+\|^{-1}\}$ for $A\in B^+(E,F)$ and $A^+\in GI(A),C_A(A^+,T)=I_F+(T-A)A^+$ and $D_A(A^+,T)=I_E+A^+(T-A)$, where I_E and I_F denote the identities on F and E, respectivelly. In 1993, Nashed M. Z. and Chen X. indicated in [1] that if $C_A^+(A^+,T)R(T)\subset R(A)$ for $T\in V(A,A^+)$, then the

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following property holds: $B=A^+C_A^{-1}(A^+,T)=D_A^{-1}(A^+,T)$ is the generalized inverse of T with $R(B)=R(A^+)$ and $N(B)=N(A^+)$. This property is essential to the theory of generalized inverse analysis. Then sevaral conditions equivalent to the property are presented (for details see [2–5]). Let T_x be an operator valued map from a topological spaces X into $B(E,F)$. Especially in 1999, the concept of locally fine point of T_x is introduced, see [2]. Thanks to it the complete rank theorem in advanced calculus and the operator rank theorem both are established, see [2,3]. The previous one gives a complete answer to the rank theorem problem presented by Beger M. S in [6], and the latter expands Penrose theorem from the case of matrices to that of operators (see [3,7] and [5]). Many applications of them are given in [5, 8, 9] and [10]. So far we may say that the generalized inverse analysis is built. In this paper we introduce the unbouded domain $\Omega(A,A^+)$ in $B(E,F)$ for $A\in B^+(E,F)$ and $A^+\in GI(A)$, and show a necessary and sufficient condition for $T\in\Omega(A,A^+)$. Moreover, several conditions equivalent to the above property for $T\in\Omega(A,A^+)$ are given in Theorem 3.1 in the next Section 3. Also we obtain a smooth diffeomorphism from $\Omega(A,A^+)$ onto itself with a fixed point A, i.e., Theorem 4.1 in Section 4 holds. Let $S=\{T\in\Omega(A,A^+):R(T)\cap N(A^+)=\{0\}\}$ for any $A\in B^+(E,F)$ and $A^+\in GI(A)$, $M(X)=\{T\in B(E,F):TN(X)\subset R(X)\}$, and $\mathcal{F}=\{M(X):\forall X\in B(E,F)\}$. Using this smooth diffeomorphism we prove that S is a smooth submanifold in $B(E,F)$ and tangent to $M(X)$ at any $X\in S$, i.e., Theorem 4.2 in the next Section 4 holds. The theorem expands the smooth integrability of $\mathcal{F}$ at A as indecated in Theorem 4.1 in [6] from a local neighborhood at A to the global unbounded domain $\Omega(A,A^+)$. These seem to be useful for developing the global analysis and geomatrical method in differential equations.

2 The domain $\Omega(A,A^+)$ in $B(E,F)$

Let $B(F)=B(F,F)$ and $B^X(F)$ be the set of all invertible operators in $B(F)$. Write $C_A(A^+,T)=I_F+(T-A)A^{-1}\in B(F)$ for $A\in B^+(E,F)$ and $A^+\in GI(A)$, where I_F denotes the identity on F. We define

$$\Omega(A,A^+)=\{T\in B(E,F):C_A(A^+,T)\in B^X(F)\}$$

for $A\in B^+(E,F)$ and $A^+\in GI(A)$. For abbreviation, write $P^{N(A)}_{R(A^+)}$, $P^{R(A^+)}_{N(A)}$, $P^{N(A^+)}_{R(A)}$ and $P^{R(A)}_{N(A^+)}$ as $P_{R(A^+)}$, $P_{N(A)}$, $P_{R(A)}$ and $P_{N(A^+)}$, respectively, in the sequel. Then $C_A(A^+,T)=P_{N(A^+)}+TA^{-1}$. We have

Theorem 2.1. *T belongs to $\Omega(A,A^+)$ if and only if the following conditions hold:*

$$N(T)\cap R(A^+)=\{0\}$$

and

$$F=R(TA^+)\oplus N(A^+). \tag{2.1}$$