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A Struwe Type Decomposition Result for a Singular Elliptic Equation on Compact Riemannian Manifolds

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Abstract. On a compact Riemannian manifold, we prove a decomposition theorem for arbitrarily bounded energy sequence of solutions of a singular elliptic equation.

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1 Introduction

Let (M,g) be an $(n\geq 3)-$dimensional Riemannian manifold. In this paper, we are interested in studying on (M,g) the asymptotic behaviour of a sequence of solutions u_α, when $\alpha\to\infty$, of the following singular elliptic equation:

$$\Delta_g u-\frac{h_\alpha}{\rho_p^2(x)}u=f(x)|u|^{2^*-2}u, \tag{E_α}$$

where $2^*=\frac{2n}{n-2}$, h_α and f are functions on M, p is a fixed point of M and $\rho_p(x)=dist_g(p,x)$ is the distance function on M based at p (see Definition 2.2).

Certainly, if the singular term $\frac{h_\alpha}{\rho_p^2(x)}$ is replaced by $\frac{n-2}{4(n-1)}Scal_g$, then equation E_α becomes the prescribed scalar curvature equation which is very known in the literature. When f is constant and the function ρ_p is of power $0<\gamma<2$, Eq. (E_α) can be seen as a case of equations that arise in the study of conformal deformation to constant scalar curvature of metrics which are smooth only in some ball $B_p(\delta)$ (see [5]).

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Equations of type (E_α) have been the subject of interest especially on the Euclidean space IR^n. Let $D^{1,2}(IR^n)$ be the Sobolev space defined as the completion of $C_o^\infty(IR^n)$, the space of smooth functions with compact support in IR^n, with respect to the norm

$$||u||^2_{D^{1,2}(IR^n)} = \int_{IR^n} |\nabla u|^2 dx.$$

A famous result has been obtained in [8] and it consists of the classification of positive solutions $u \in D^{1,2}(IR^n)$ of the equation

$$\Delta u - \frac{\lambda}{|x|^2} u = u^{\frac{n+2}{n-2}}, \tag{E}$$

where $0<\lambda<\frac{(n-4)^2}{4}$, into the family of functions

$$u_\lambda(x) = C_\lambda \left(\frac{|x|^{a-1}}{1+|x|^{2a}} \right)^{\frac{n}{2}-1},$$

where C_λ is some constant and $a=\sqrt{1-\frac{4\lambda}{(n-2)^2}}$.

In terms of decomposition of Palais-Smale sequences of the functional energy, this family of solutions was employed in [6] to construct singular bubbles,

$$\mathcal{B}_\lambda^{\varepsilon_\alpha, y_\alpha} = \varepsilon_\alpha^{\frac{2-n}{2}} u_\lambda \Big(\frac{x-y_\alpha}{\varepsilon_\alpha} \Big) \quad \text{with} \quad \frac{|y_\alpha|}{\varepsilon_\alpha} \to 0,$$

which, together with the classical bubbles caused by the existence of critical exponent

$$\mathcal{B}_0^{\varepsilon_\alpha, y_\alpha} = \varepsilon_\alpha^{\frac{2-n}{2}} u_0 \Big(\frac{x-y_\alpha}{\varepsilon_\alpha} \Big) \quad \text{with} \quad \frac{|y_\alpha|}{\varepsilon_\alpha} \to \infty,$$

where u_0 being the solution of the non perturbed equation $\Delta u = u^{\frac{n+2}{n-2}}$, give a whole picture of the decomposition of the Palaise-Smale sequences. This decomposition result has been proved in [6] and was the key component for the obtention of interesting existence results for Eq. (E) with a function K get involved in the nonlinear term. Similar decomposition result has been obtained in [1] for Eq. (E) with small perturbation, the authors described asymptotically the associated Palais-Smale sequences of bounded energy.

The compactness result obtained in this paper can be seen as an extension to Riemannian context of those obtained in [6] and [1] in the Euclidean context, the difficulties when working in the Riemannian setting reside mainly in the construction of bubbles.

Historically, a famous compactness result for elliptic value problems on domains of $\mathbb{R}^n$ has been obtained by M. Struwe in [7]. Struwe's result has been extended later by O. Druet et al. in [2] to elliptic equations on Riemannian manifolds in the form

$$\Delta_g u + h_\alpha u = u^{2^*-1}.$$