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A FEM for Solving Two-Dimensional Nonlinear Elliptic-Parabolic Interface Problems with Nonhomogeneous Jump Conditions

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Abstract. In this paper, a new method was proposed for solving two-dimensional nonlinear elliptic-parabolic interface problems with nonhomogeneous jump conditions. The method we used is a finite element method coupled with Newton's method. It is very simple and easy to implement. The grid we used here is body-fitting grids based on the idea of semi-Cartesian grid. Numerical experiments show that this method is about second order accurate in the L^∞ norm for different kinds of nonlinear terms and interface with complicated geometry.

AMS subject classifications: 65N30

Key words: Finite element method, nonlinear elliptic-parabolic interface problems, nonhomogeneous jump conditions.

1 Introduction

Interface problems arise naturally in a wide variety of applications when two or more materials meet. In the past three decades, much attention has been paid to the numerical solution of elliptic equations with discontinuous coefficients and singular sources since the pioneering work of Peskin [14] on the first order accurate immersed boundary method.

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Motivated by the immersed boundary method, in [15], the immersed interface method (IIM) was presented. This method achieves second order accuracy by incorporating the interface conditions into the finite difference stencil in a way that preserves the interface conditions in both solution and its flux.

In [11], the immersed finite element method (IFE) is introduced for interface problems with homogeneous jump conditions. The main idea is that the test and trial function basis are continuous but not smooth across the interface in an interface triangle. In [12], the method is generalized to deal with nonhomogeneous flux jump condition. In [13], the partially penalized IFE method is introduced to penalize the discontinuity of the IFE function on neighboring interface triangles.

In [16], a non-traditional finite element formulation for solving elliptic equations with smooth or sharp-edged interfaces was proposed with non-body-fitting grids for $[u]\neq 0$ and $[\beta u_n]\neq 0$. It achieved second order accuracy in the L^∞ norm for smooth interfaces and about 0.8th order for sharp-edged interfaces. In [17], the method is modified and improved to close to 2nd order accurate for sharp-edged interfaces, and it is extended to handle general elliptic equations with matrix coefficient and lower order terms. In [18–20], the method is further analyzed and implemented in three dimension as well. The resulting linear system is non-symmetric but positive definite.

In [21], the matched interface and boundary (MIB) method was proposed to solve elliptic equations with smooth interfaces. In [22], the MIB method was generalized to treat sharp-edged interfaces. With an elegant treatment, second order accuracy was achieved in the L^∞ norm. In [23], the MIB method was extended to treat three dimensional interface problems. Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [24] for two dimensional problems and [25] for three dimensional problems. Another class of methods is the Boundary Condition Capturing Method [26–28].

Nonlinear elliptic-parabolic interface problem is a more challenging problem compared with linear elliptic interface problems. It appears in electromagnetism when we study the enzyme fermentation reactor [1] and in the heat-mass transfer when the heat flux goes through a domain with different subdomains [3,4]. It is also applied to describe the fluid flow in partially saturated media [2], in which an elliptic equation is employed to simulate the fluid flow in the saturated region and a parabolic equation is employed to simulate the flow in the unsaturated region. In [5], a numerical method is introduced to solve the one-dimensional nonlinear elliptic-parabolic interface problem with homogeneous jump conditions. In this paper, we present a new method to solve this nonlinear elliptic-parabolic interface problem in two-dimension with nonhomogeneous jump conditions. This is a non-trivial extension to the one dimensional problem. The jump conditions are nonhomogeneous, making it even more challenging. The nonlinear term in the equation is another challenging issue. Extensive numerical experiments shows that our method is robust for this elliptic-parabolic interface problem with different kinds of nonlinear terms, but also stable for the problem with nonhomogeneous jump conditions and complicated geometry interface. It can achieve about second order accuracy in the