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A New Fictitious Domain Method for Elliptic Problems with the Third Type Boundary Conditions

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Abstract. In this article, we discuss a modified least–squares/fictitious domain method for the solution of linear elliptic boundary value problems with the third type of boundary conditions (Robin boundary conditions). Let Ω and ω be two bounded domains of $\mathbb{R}^d$ such that $\overline{\omega}\subset\Omega$. For a linear elliptic problem in $\Omega\backslash\overline{\omega}$ with Robin boundary conditions on the boundary γ of ω, we accelerate the original least–squares/fictitious domain method in Glowinski & He [1] and present a modified least–squares formulation. This method is still a virtual control type and relies on a least-squares formulation, which makes the problem solvable by a conjugate gradient algorithm operating in a well chosen control space. Numerical results show that our method costs much less iterations and the optimal order of convergence is obtained.

AMS subject classifications: 65M85, 65N85, 76M10, 93E24

Key words: Least–squares methods, fictitious domain methods, finite element methods, Robin boundary conditions.

1 Introduction

Fictitious domain methods have been developed for a long time and have known many realizations. They have enjoyed many denominations. To the best of our knowledge, these methods have been introduced by Hyman [2] and further investigated by many authors; let us mention, among others, Saul'ev [3, 4] and Buzbee, Dorr, George and Golub [5]. In Glowinski, Pan and Periaux [6–8] and Glowinski, Pan, Kearsley and Periaux [9], fictitious domain methods were discussed for the solution of Dirichlet problems. The Dirichlet boundary is enforced as a side constraint, using a boundary supported Lagrange multiplier. Recently, Wang and Sun developed a fictitious domain method with distributed Lagrange multiplier for parabolic problems with moving interfaces, see [10]. Examples of non-Lagrange multiplier based fictitious domain methods can be

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found in the immersed boundary method of C. Peskin and his collaborators (see, e.g., Peskin [13, 14] and Peskin and Mcqueen [15]) for the simulation of incompressible viscous flow in regions with elastic moving boundaries. A boundary condition-enforced immersed boundary method is introduced into the Lattice Boltzmann method in [16]. Application of immersed interface methods can be found in [17] and [18], which is still regarded as one kind of fictitious domain method. A (brief) history of fictitious domain methods can be found in, e.g., [12, Chapter 8].

The ideology of fictitious domain methods is to extend a problem initially posed on a geometrically complex shaped domain to a larger simpler domain, which has two main advantages when constructing numerical schemes:

(i) The extended domain is geometrically simpler and allows the use of fast solvers.

(ii) The same fixed mesh can be used for the entire computation, eliminating thus the need for repeated re–meshing and projection.

When we study the simulation of flow past a cylinder, the fictitious domain method with boundary–supported Lagrangian multiplier works well. When we study the motion of particulate flow, the fictitious domain method with volume–supported Lagrange multiplier is efficient. All the above consider no–slip boundary conditions at the interface between fluid and rigid bodies. The Navier slip boundary conditions or generalized Navier slip boundary conditions have been verified in micro–fluidics research work. The present Lagrange multiplier based fictitious domain methods discussed in [6–12], which rely on H^1–extensions, are not easy to generalize to the slip situation. Therefore, a new fictitious domain method needs to be proposed for problems with slip boundary conditions. The authors in [1] discuss the solution of linear elliptic boundary value problems with Robin boundary conditions, which introduced a least-squares formulation to solve an exact controllability problem. Numerical results showed that it suggested optimal order of convergence but with many iteration numbers.

In this article, we are going to develop a modified least-squares formulation which can accelerate solving the solution of linear elliptic problems with Robin boundary conditions; we see this as one important step to figure out a new fictitious domain method suited to slip boundary conditions. As similar in [1], the modified formulation is still virtual control type in the sense of J. L. Lions; see [19]. The conjugate gradient algorithm can be used in a well-chosen control space.

The rest of this article is organized as follows. The formulation of the boundary value problems is proposed in Section 2. We describe a modified least–squares/fictitious domain method for the solution of linear elliptic problems with the third type boundary conditions in Section 3. In Section 4, we discuss the conjugate gradient solution of the least–squares problems introduced in Section 3. The finite element implementation of the above methodology is discussed in Section 5. Finally, we present in Section 6 the results of numerical experiments; in particular, these results suggest optimal order of convergence for various norms of the approximation error and much less iteration numbers.