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# Convergence Rates of a Class of Predictor-Corrector Iterations for the Nonsymmetric Algebraic Riccati Equation Arising in Transport Theory

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**Abstract.** In this paper, we analyse the convergence rates of several different predictor-corrector iterations for computing the minimal positive solution of the nonsymmetric algebraic Riccati equation arising in transport theory. We have shown theoretically that the new predictor-corrector iteration given in [Numer. Linear Algebra Appl., 21 (2014), pp. 761–780] will converge no faster than the simple predictor-corrector iteration and the nonlinear block Jacobi predictor-corrector iteration. Moreover the last two have the same asymptotic convergence rate with the nonlinear block Gauss-Seidel iteration given in [SIAM J. Sci. Comput., 30 (2008), pp. 804–818]. Preliminary numerical experiments have been reported for the validation of the developed comparison theory.

**AMS subject classifications**: 65F50, 15A24

**Key words**: Convergence rate, predictor-corrector iterations, nonsymmetric algebraic Riccati equation, regular splitting.

## 1 Introduction

We consider the nonsymmetric algebraic Riccati equation (NARE) arising in transport theory [10–12]

$$\mathcal{R}(X)=XCX-AX-XD+B=0, \tag{1.1}$$

where coefficient matrices are of forms

$$A=\Delta-eq^T,\quad B=ee^T,\quad C=qq^T,\quad D=\Gamma-qe^T,$$

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with

$$e=(1,1,\cdots,1)^T,\qquad q=(q_1,q_2,\cdots,q_n)^T,\qquad q_i=\frac{c_i}{2\omega_i},$$
$$\Delta=\mathrm{Diag}(\delta_1,\delta_2,\cdots,\delta_n),\qquad \Gamma=\mathrm{Diag}(\gamma_1,\gamma_2,\cdots,\gamma_n),$$
$$\delta_i=\frac{1}{c\omega_i(1+\alpha)},\quad \gamma_i=\frac{1}{c\omega_i(1-\alpha)},\quad i=1,2,\cdots,n,$$

and $\alpha\in[0,1)$, $c\in(0,1]$. The two parameter sets $\{\omega_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$ denote the nodes and weights, respectively, of the Gauss-Legendre formula satisfying

$$0<\omega_n<\cdots<\omega_1<1,\quad \sum_{i=1}^n c_i=1,\quad c_i>0.$$

The minimal positive solution of the NARE (1.1) is of great interest in physics. The existence of the minimal positive solution has been well studied in [10, 12]. It is shown in [14] that the minimal positive solution $X^*$ of (1.1) has the form

$$X^*=T\circ(u^*(v^*)^T).$$

Here the symbol "$\circ$" is the Hadamard product, $T=(t_{ij})$ with $t_{ij}=\frac{1}{\delta_i+\gamma_j}$ and $(u^*,v^*)$ is the minimal positive solution of the vector equations

$$\begin{cases} u=u\circ(Pv)+e,\\ v=v\circ(Qu)+e,\end{cases}\tag{1.2}$$

where $P$ and $Q$ are $n\times n$ positive matrices with their respective $(i,j)$ element

$$p_{ij}=\frac{q_j}{\delta_i+\gamma_j}\quad\text{and}\quad q_{ij}=\frac{q_j}{\delta_j+\gamma_i}.$$

Let $A$, $B$ be $n\times n$ real matrices, throughout this paper we write $A>B$ (or $A\geq B$) by meaning that all elements in $A$ are greater than (or greater than and equal to) those in $B$. For $n\times n$ real matrices $K$, $M$ and $N$, $K=M-N$ is called a *regular splitting* of the matrix $K$ if $M$ is nonsingular with $M^{-1}\geq 0$ and $N\geq 0$ [20, Definition 3.28].

By noting the special structures in (1.2), several fixed-point iterative methods including the SI method, the MSI method, the NBJ method and the NBGS method have been proposed in [1,2,18] for computing the minimal positive solution of the vector equations (1.2). Their corresponding iterative schemes are all of $\mathcal{O}(n^2)$ complexity per iteration and could be viewed as coming from various regular splittings of the $M$-matrix

$$K=\begin{pmatrix} I-\mathrm{diag}(Pv^*) & -\mathrm{diag}(u^*)P\\ -\mathrm{diag}(v^*)Q & I-\mathrm{diag}(Qu^*)\end{pmatrix},\tag{1.3}$$