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# Uniform Convergence Analysis of a Higher Order Hybrid Stress Quadrilateral Finite Element Method for Linear Elasticity Problems

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**Abstract.** This paper derives a higher order hybrid stress finite element method on quadrilateral meshes for linear plane elasticity problems. The method employs continuous piecewise bi-quadratic functions in local coordinates to approximate the displacement vector and a piecewise-independent 15-parameter mode to approximate the stress tensor. Error estimation shows that the method is free from Poisson-locking and has second-order accuracy in the energy norm. Numerical experiments confirm the theoretical results.

**AMS subject classifications**: 65N12, 65N30

**Key words**: Linear elasticity, hybrid stress finite element, Poisson-locking, second-order accuracy.

## 1 Introduction

Assumed stress hybrid finite element method (also called hybrid stress finite element method), based on Hellinger-Reissner variational principle, is known to be an efficient approach to improve the performance of the standard bilinear quadrilateral (or trilinear hexahedral) displacement finite element in the analysis of elasticity problems, cf. [6–10, 12–19, 21, 23, 24].

The first work on the construction of assumed stress hybrid method is due to Pian [6], where the assumed stress field is assumed to satisfy homogenous equilibrium equations pointwise. In [7] Pian and Chen proposed a new type of hybrid method by enhancing

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the stress equilibrium equations in a variational sense and by adopting the natural coordinates for the stress approximation. Later Pian and Sumihara [8] derived, through a rational choice of stress terms, a robust 4-node hybrid stress quadrilateral element (abbr. PS element) which yields uniformly accurate results at the numerical benchmark tests. In [17] Xie and Zhou constructed an accurate 4-node hybrid stress quadrilateral finite element (called ECQ4 element) by optimizing stress modes with a so-called energy compatibility condition [23]. Zhou and Xie [24] gave a unified convergence analysis for three types of 4-node hybrid stress/strain finite elements, including PS element, but the upper bound of the error estimate is not uniform in the Lamé constant $\lambda$. In [20] Yu, Xie and Carstensen analyzed the hybrid stress methods of PS and ECQ4, and obtained uniform convergence and a posteriori error estimation.

We note that, due to the use of low order displacement approximations such as the bilinear interpolation, etc., the 4-node hybrid stress quadrilateral elements mentioned above are of at most first-order accuracy. To the authors' best knowledge, so far there is few literature considering higher order hybrid stress quadrilateral finite elements.In this paper, we shall construct a higher-order hybrid stress quadrilateral element for 2D linear elasticity problems by using continuous piecewise bi-quadratic interpolations for the displacement approximation and a piecewise independent 15-parameter mode for the stress approximation. Since the stress parameters can be eliminated at the element level, the final discrete system only has nodal displacements as unknowns, and the computational cost of the proposed element is almost as same as that of bi-quadratic ($Q_2$) displacement element. We will show that the element is of second-order accuracy in the energy norm and free from Poisson-locking in the sense that the error bound in the a priori error estimate is independent of the Lamé constant $\lambda$.

Throughout the paper, we use notation $a \lesssim b$ (or $a \gtrsim b$) to represent that there exists a constant $C$ independent of mesh size $h$ and the Lamé constant $\lambda$ such that $a \leq Cb$ (or $a \geq Cb$), and use $a \approx b$ to denote $a \lesssim b \lesssim a$.

The rest of this paper is organized as following. In Section 2, we describe the model problem and weak formulations. Section 3 shows the construction of the higher order hybrid stress element. Section 4 is devoted to the uniform stability analysis and a priori error estimation. The final section provides some numerical results.

## 2 Weak formulations

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where $\text{meas}(\Gamma_D) > 0$. We consider the linear plane elasticity problem

$$\begin{cases} -\text{div}\sigma = f & \text{in } \Omega, \\ \sigma = \mathbb{C}\epsilon(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \sigma n = g & \text{on } \Gamma_N, \end{cases} \tag{2.1}$$