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Existence and Asymptotic Behavior of Positive Solutions for Variable Exponent Elliptic Systems

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Abstract. In this paper, our main purpose is to establish the existence of positive solution of the following system

$$\begin{cases} -\Delta_{p(x)}u=F(x,u,v), & x\in\Omega, \\ -\Delta_{q(x)}v=H(x,u,v), & x\in\Omega, \\ u=v=0, & x\in\partial\Omega, \end{cases}$$

where $\Omega=B(0,r)\subset\mathbf{R}^N$ or $\Omega=B(0,r_2)\setminus\overline{B(0,r_1)}\subset\mathbf{R}^N$, $0<r$, $0<r_1<r_2$ are constants. $F(x,u,v)=\lambda^{p(x)}[g(x)a(u)+f(v)]$, $H(x,u,v)=\theta^{q(x)}[g_1(x)b(v)+h(u)]$, $\lambda,\theta>0$ are parameters, $p(x),q(x)$ are radial symmetric functions, $-\Delta_{p(x)}=-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. We give the existence results and consider the asymptotic behavior of the solutions. In particular, we do not assume any symmetric condition, and we do not assume any sign condition on $F(x,0,0)$ and $H(x,0,0)$ either.

AMS subject classifications: 35J60, 35J62

Key words: Positive solution, $p(x)$-Laplacian, asymptotic behavior, sub-supersolution.

1 Introduction

In this paper, our main purpose is to establish the existence of positive solution of the following system

$$\begin{cases} -\Delta_{p(x)}u=F(x,u,v), & x\in\Omega, \\ -\Delta_{q(x)}v=H(x,u,v), & x\in\Omega, \\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega=B(0,r)\subset\mathbf{R}^N$ or $\Omega=B(0,r_2)\setminus\overline{B(0,r_1)}\subset\mathbf{R}^N$, r and $r_1<r_2$ are positive constants, $F(x,u,v)=\lambda^{p(x)}[g(x)a(u)+f(v)]$, $H(x,u,v)=\theta^{q(x)}[g_1(x)b(v)+h(u)]$ and $p(x),q(x)\in C^1(\overline{\Omega})$ are radial symmetric positive functions, i.e., $p(x)=p(|x|)$, $q(x)=q(|x|)$, the operator $-\Delta_{p(x)}=-\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian and the corresponding equation is called a variable exponent equation.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It aries from nonlinear elasticity theory, electro-rheological fluids, etc. (see [17, 27]). Many results have been obtained on this kind of problems, for example [1–3, 5–7, 9, 13]. On the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [1, 3, 5]. For the existence results for the elliptic problems with variable exponents, we refer to [7, 13, 21–24].

For the special case, $p(x)\equiv p$ (a constant), (1.1) becomes the well known p-Laplacian system. There have been many papers on this class of problems, see [4, 12, 19] and the reference therein. We point out that elliptic equations involving the $p(x)$-Laplacian are not trivial generalizations of similar problems studied in the constant case, since the $p(x)$-Laplacian operator is nonhomogeneity. Thus, some techniques which can be applied in the case of the p-Laplacian operators will fail in that new station, such as the Lagrange Multiplier Theorem. Another example is that, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)}=\inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}}\frac{\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx}{\int_\Omega\frac{1}{p(x)}|u|^{p(x)}dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)}>0$ (see [11]). But the facts that the first eigenvalue $\lambda_p>0$ and the existence of the first eigenfunction are very important in the study of p-Laplacian problems. There are more difficulties in discussing the existence and asymptotic behavior of solutions of variable exponent problems.

In [12], the authors studied the existence of positive weak solutions for the following problem:

$$\begin{cases}-\Delta_p u=\lambda f(v), & x\in\Omega,\\ -\Delta_p v=\lambda g(u), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega.\end{cases}\tag{1.2}$$

Under the condition of

$$\lim_{s\to\infty}\frac{f(M[g(s)]^{\frac{1}{p-1}})}{s^{p-1}}=0,\quad\forall M>0,\tag{1.3}$$

the authors gave the existence of positive solutions for problem (1.2).

In [4], the author considered the existence and nonexistence of positive weak solu-