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The Eigenfunctions and Exact Solutions of Discrete mKdV Hierarchy with Self-Consistent Sources via the Inverse Scattering Transform

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Abstract. Another form of the discrete mKdV hierarchy with self-consistent sources is given in the paper. The self-consistent sources is presented only by the eigenfunctions corresponding to the reduction of the Ablowitz-Ladik spectral problem. The exact soliton solutions are also derived by the inverse scattering transform.

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Key words: Discrete mKdV hierarchy with self-consistent sources, eigenfunction, exact solutions, inverse scattering transform.

1 Introduction

Soliton equations with self-consistent sources model some phenomena in hydrodynamics, plasma physics, solid state physics [1–9]. For reason of an arbitrary time-dependence in dispersion relation, the self-consistent sources result in some interesting dynamical characters [10, 11]. In terms of solutions, some methods have been used to investigate some soliton equations including those soliton equations with self-consistent sources, such as the inverse scattering transform (IST), Bäcklund transformation, Darboux transformation, bilinear method [12–23]. Recently, the transformed rational function method, the multiple exp-function algorithm, the linear superposition principle and the invariant subspace method are also developed to solve some soliton equations [24–27].

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Ablowitz-Ladik (A-L for short) spectral problem [28–30] attracts much attention. Usually, one eigenvalue problem admits a pair of linear problem, for which the hierarchy of soliton equation is just the compatible condition. For more details, one can see for example [31–35] and references therein. Recently, the discrete mKdV hierarchy with self-consistent sources is investigated by constructing non-auto-Bäcklund transformations [19], where the self-consistent sources are expressed by both of the eigenfunctions and adjoint eigenfunctions corresponding to the A-L spectral problem and its adjoint problem, respectively. However, there exists some relationship between the eigenfunctions and the adjoint eigenfunctions, which implies that the sources can be expressed only by the eigenfunctions. This relationship leads to another form of the discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy), which is different from the hierarchy in [19,32].

The paper is organized as follows. In Section 2, we briefly deduce the discrete mKdV hierarchy with self-consistent sources. In Section 3, we describe the relationship between the eigenfunctions and the adjoint eigenfunctions, and deduce another form of the discrete mKdV hierarchy with self-consistent sources. In Section 4, we investigate the soliton solutions of DmKdVSCS hierarchy through the IST. Some dynamics are given.

2 The discrete mKdV hierarchy with self-consistent sources

In this section, we derive the discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy). Our approach is a little different from the one using constrained flows [19,32].

We denote the function $f_n = f(n,t)$ with $n \in \mathbb{Z}$. The shift operator E is defined by $Ef_n = f_{n+1}$ for arbitrary function f_n. By Δ we denote $E-1$. The A-L spectral problem is known as [30]

$$\begin{pmatrix} \phi_{1,n+1} \\ \phi_{2,n+1} \end{pmatrix} = \begin{pmatrix} z & Q_n \\ R_n & 1/z \end{pmatrix} \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}. \tag{2.1}$$

Let us begin with A-L spectral problem in the case of $R_n = \varepsilon Q_n$ $(\varepsilon = \pm 1)$,

$$\begin{pmatrix} \phi_{1,n+1} \\ \phi_{2,n+1} \end{pmatrix} = \boldsymbol{M}_n \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \quad \boldsymbol{M}_n = \begin{pmatrix} z & Q_n \\ \varepsilon Q_n & 1/z \end{pmatrix}, \tag{2.2a}$$

and the time evolution

$$\begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}_t = \boldsymbol{U}_n \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \quad \boldsymbol{U}_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}. \tag{2.2b}$$

The adjoint spectral problem of (2.2a) is

$$\begin{pmatrix} \varphi_{1,n-1} \\ \varphi_{2,n-1} \end{pmatrix} = \boldsymbol{M}_n^T \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \end{pmatrix}, \tag{2.3}$$