Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **7**, No. 3, pp. 323-337

DOI: 10.4208/aamm.2014.m615
June 2015

Superconvergence of bi-k Degree Time-Space Fully Discontinuous Finite Element for First-Order Hyperbolic Equations

Hongling Hu and Chuanmiao Chen*

College of Mathematics and Computer Science, Key Laboratory of High Performance Computing and Stochastic Information Processing, Hunan Normal University, Changsha 410081, China

Received 10 May 2014; Accepted (in revised version) 22 August 2014

Abstract. In this paper, we present a superconvergence result for the bi-k degree time-space fully discontinuous finite element of first-order hyperbolic problems. Based on the element orthogonality analysis (EOA), we first obtain the optimal convergence order of discontinuous Galerkin finite element solution. Then we use orthogonality correction technique to prove a superconvergence result at right Radau points, which is higher one order than the optimal convergence rate. Finally, numerical results are presented to illustrate the theoretical analysis.

AMS subject classifications: 65M60, 35L04, 65M12

Key words: Superconvergence, hyperbolic equation, finite element, discontinuous Galerkin method, Radau points.

1 Introduction

Hyperbolic partial differential equations arise in a broad spectrum of disciplines where wave motion or advective transport is important: gas dynamics, geophysics, optics and so on. In 1950s, Friedrichs studied the first order symmetric positive definite hyperbolic systems, and proved the solvability of the initial value problem using the energy method. Since that hyperbolic problems have gained significant attention. When using continuous finite element methods for first-order hyperbolic problems, because the bilinear form is not symmetric, the convergence analysis is very difficult.

The Discontinuous Galerkin method (DG) was first introduced in 1973 by Reed and Hill [1], in the framework of neutron linear transport. Subsequently, Lesaint and

*Corresponding author.
Email: hhling625@163.com (H. L. Hu), cmchen@hunnu.edu.cn (C. M. Chen)

Raviart [2] solved the neutron transport equation which is time independent linear hyperbolic equation, and proved that the DG method is convergent with the optimal order of accuracy, namely $\mathcal{O}(h^{k+1})$ in the sense of L^2 norm. In [3], Johnson and Pitkaranta applied the DG method to a scalar linear hyperbolic equation and proved L^2 error estimates of $\mathcal{O}(h^{k+1/2})$ for arbitrary meshes when the solution is sufficiently smooth. The literature [5] indicated this estimate is actually sharp for most general situations. However the optimal $\mathcal{O}(h^{k+1})$ error bound can be proved in most cases, and always be observed from numerical experiments [4,11].

However, due to difficulties in studying time-space fully discontinuous finite element, lots of existing work, e.g., those mentioned above, focused on semi-discrete schemes, or the fully discrete schemes in which the DG discretization is only used for the spatial variables, and the time discretization is achieved by the difference method. The most famous is the Runge-Kutta discontinuous Galerkin (RKDG) method, namely the DG discretization is used for the spatial variables, and explicit, nonlinearly stable high order Runge-Kutta methods are used to discretize the time variable. Using RKDG, from 1980s, Cockburn and Shu et al. made a series of excellent work on hyperbolic conservation laws [6–10]. In [13], Cheng and Shu obtained $\mathcal{O}(h^{k+1+\alpha})$ with $\alpha=1/2$ for linear, time-dependent hyperbolic equations in one-dimension, with uniform meshes and periodic boundary conditions. Then, this result was generalized to general polynomial degree k, on non-uniform regular meshes, and without periodic boundary conditions [14]. The result in [14] was further improved to $\alpha=1$ in [16]. In [15], the superconvergence with $\alpha=1/2$ was proved for scalar nonlinear conservation laws with a fixed wind direction in one space dimension. Adjerid et al. [19] proved that the DG solution of the ordinary differential equation is superconvergent of order $2k+1$ at the downwind end of each element while maintaining an order of $k+2$ at the remaining Radau points. Later, these results were extended to two-dimensional problems on rectangular meshes [20] and nonlinear hyperbolic problems [21]. There are also many superconvergence results on hyperbolic problems, for example, negative-order norm error estimates [12,17], posteriori error estimates [22,25], linear symmetrizable hyperbolic systems [23,24] and so on. Zhang [26] discussed the superconvergence by using bilinear discontinuous finite elements on a rectangular domain, and obtained an $\mathcal{O}(h^2)$ order superconvergence error estimate under the conditions of almost uniform partition and the H^3 regularity for the exact solutions. For a more complete list of citations on the DG methods and the superconvergence of hyperbolic equations, we can see [18] and references therein.

What mentioned above deals with the semi-discrete schemes or the fully schemes with the temporal discretization by Runge-Kutta method. In practice, the time discretization also plays an important role. However, to the best of our knowledge, very few superconvergence results about the time-space fully discontinuous schemes have been obtained for hyperbolic problems, even in the one-dimensional case. In this paper, we will study the bi-k degree time-space fully discontinuous finite element approximation for first order hyperbolic equations on rectangular mesh, and will obtain some superconvergence results.