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# Superconvergence of Finite Element Methods for Optimal Control Problems Governed by Parabolic Equations with Time-Dependent Coefficients

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**Abstract.** In this article, a fully discrete finite element approximation is investigated for constrained parabolic optimal control problems with time-dependent coefficients. The spatial discretisation invokes finite elements, and the time discretisation a nonstandard backward Euler method. On introducing some appropriate intermediate variables and noting properties of the $L^2$ projection and the elliptic projection, we derive the superconvergence for the control, the state and the adjoint state. Finally, we discuss some numerical experiments that illustrate our theoretical results.

**AMS subject classifications**: 35B37, 49J20, 65N30

**Key words**: Superconvergence, finite element methods, optimal control problems, parabolic equations, interpolate operator.

## 1. Introduction

Optimal control problems arise extensively in many social, economic, scientific and engineering applications. Nowadays, finite element (FE) methods seem to be the most widely used numerical approach for solving optimal control problems — cf. Refs. [1, 6, 8, 11, 20, 25, 28, 29] for systematic discussions of such methods for PDE and optimal control problems.

In the FE approximation for elliptic optimal control problems, *a priori* error estimates have been established [15], *a posteriori* error estimates of both recovery and residual type derived [13, 18], and adaptive FE approximations for optimal control problems investigated [12]. Recently, *a posteriori* and *a priori* error estimates in mixed FE methods for elliptic optimal control problems have been obtained in Refs. [5] and [30], respectively;

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and a variational discretisation approximation for optimal control problems with control constraints has also been considered [9,10].

Parabolic optimal control problems are frequently met in mathematical models describing petroleum reservoir simulation, environmental modelling, groundwater contaminant transport, and many other applications that can be difficult to handle. *A priori* and *a posteriori* error estimates of FE methods for such optimal control problems have also been established in Refs. [16] and [19,31], respectively. Relevant *a priori* estimates for space-time FE discretisation have also recently been obtained [22,23], and a characteristic FE approximation for optimal control problems governed by transient advection-diffusion equations has been investigated [7].

There has been extensive research on the superconvergence of FE methods for elliptic optimal control problems. The superconvergence of linear, semilinear and bilinear elliptic optimal control problems was established in Refs. [24], [3] and [32], respectively; and for mixed FE approximation of Stokes optimal control problems in Ref. [17]. Some superconvergence results of mixed FE approximation for elliptic optimal control problems have also been obtained [2,4,21,33]. Recently, we have derived the superconvergence of fully discrete FE approximation for linear and semilinear parabolic control problems in Refs. [27] and [26], respectively. In this article, we investigate the superconvergence of fully discrete FE methods for an optimal control problem governed by parabolic equations with time-dependent coefficients and control constraints, and then undertake some numerical experiments to confirm our theoretical results.

We consider the following parabolic optimal control problem:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\int_\Omega (y-y_d)^2 + \int_\Omega u^2\right) dt\,, \\ y_t(x,t) - \operatorname{div}(A(x,t)\nabla y(x,t)) = f(x,t) + u(x,t)\,, \quad x\in\Omega,\ t\in J\,, \\ y(x,t) = 0\,, \quad x\in\partial\Omega,\ t\in J\,, \\ y(x,0) = y_0(x)\,, \qquad x\in\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, $J=[0,T]$ with $T>0$, the coefficient $A(x,t)=(a_{ij}(x,t))_{2\times 2}\in \left(W^{1,\infty}(J;W^{1,\infty}(\bar\Omega))\right)^{2\times 2}$ such that for any $\xi\in\mathbb{R}^2$ and $t\in J$, $(A(x,t)\xi)\cdot\xi \ge c\mid\xi\mid^2$ with $c>0$. Let $f(x,t)\in C(J;L^2(\Omega))$ and $y_0(x)\in H_0^1(\Omega)$. Further more, we assume that $K$ is a nonempty closed convex set in $L^2(J;L^2(\Omega))$ defined by

$$K=\left\{\, v(x,t)\in L^2(J;L^2(\Omega)) : a\le v(x,t)\le b, \quad a.e.\ \ (x,t)\in\Omega\times J \,\right\},$$

where $a$ and $b$ are constants.

Here we also adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on $\Omega$ with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$, set $H_0^1(\Omega)\equiv\left\{v\in H^1(\Omega): v|_{\partial\Omega}=0\right\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, and denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of all $L^s$ integrable functions from $J$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}dt)^{\frac{1}{s}}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, one can define the space