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A Partially Penalised Immersed Finite Element Method for Elliptic Interface Problems with Non-Homogeneous Jump Conditions

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Abstract. A partially penalised immersed finite element method for interface problems with discontinuous coefficients and non-homogeneous jump conditions based on unfitted meshes independent of the interface is proposed. The arising systems of linear equations have symmetric positive definite matrices which allows the use of fast solvers and existing codes. Optimal error estimates in an energy norm are derived. Numerical examples demonstrate the efficiency of the method.

AMS subject classifications: 65N15, 65N30, 35J60

Key words: Immersed finite element method, interface problem, Cartesian mesh, non-homogeneous jump condition, closest-point projection.

1. Introduction

We consider an immersed interface finite element method to solve the elliptic interface problem

$$-\nabla\cdot\beta(x,y)\nabla u(x,y)=f(x,y), \qquad x\in\Omega\backslash\Gamma, \tag{1.1}$$

$$u(x,y)=0, \qquad x\in\partial\Omega, \tag{1.2}$$

together with the following non-homogeneous jump conditions across the interface Γ:

$$[u]_\Gamma=u^+-u^-=w, \tag{1.3}$$

$$\left[\beta\frac{\partial u}{\partial \mathbf{n}}\right]_\Gamma=\beta^+\nabla u^+\cdot\mathbf{n}-\beta^-\nabla u^-\cdot\mathbf{n}=Q, \tag{1.4}$$

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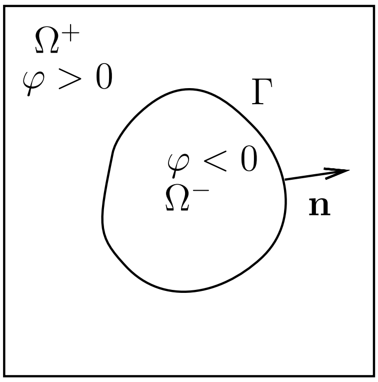

Figure 1: The geometry of an interface problem.

where $u^{\pm}(x,y) = u(x,y)|_{\Omega^{\pm}}$ and $\mathbf{n}$ is a unit normal to the interface pointing from Ω^- to Ω^+. In Fig. 1 for an illustration, without loss of generality, we assume that $\Omega \subset \mathbb{R}^2$ is a rectangular domain, the interface is a closed and smooth curve separating Ω into two subdomains Ω^-, Ω^+ such that Ω^- lies strictly inside Ω, and the coefficient $\beta(x,y)$ is a positive and piecewise constant function — i.e.

$$\beta(x,y) = \begin{cases} \beta^-, & (x,y) \in \Omega^-, \\ \beta^+, & (x,y) \in \Omega^+, \end{cases} \qquad \text{and} \qquad \beta^{\pm} > 0\,.$$

The homogeneous Dirichlet boundary condition $u = 0$ is used just for convenience in the theoretical analysis, since we focus on the interface. Indeed, other boundary conditions (e.g. non-homogeneous boundary conditions or Neumann boundary conditions) can be treated using standard finite element techniques. As in the classical level set method, the interface Γ is implicitly defined as the zero level set of a smooth function $\varphi(x,y)$ satisfying

$$\varphi(x,y) \begin{cases} <0\,, & (x,y) \in \Omega^-, \\ =0\,, & (x,y) \in \Gamma\,, \\ >0\,, & (x,y) \in \Omega^+. \end{cases}$$

The unit normal $\mathbf{n}$ to the interface Γ pointing from Ω^- to Ω^+ is then $\mathbf{n} = \nabla\varphi/\|\nabla\varphi\|_2$. It is notable that the method we propose is not restricted to the level set representation of the interface, for it also works for a parametric representation of the interface — cf. Section 2.4.

The interface problem (1.1)-(1.4) arises in many important scientific and engineering applications. For example, in electrostatic field computations this interface problem involves interface data Q referring to surface charge density [14]. For the computation of the temperature in composite media with thermal contact resistance effect, the solution may have jumps across the interface — i.e., $w \neq 0$. We are interested in solving such an interface problem by a finite element (FE) method based on unfitted meshes independent of the interface. Numerical methods using such meshes have some advantages compared with body fitted meshes, including the attractive ease in handling problems with moving interfaces. An early numerical method that used an unfitted mesh is