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Spectral Direction Splitting Schemes for the Incompressible Navier-Stokes Equations

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Abstract. We propose and analyze spectral direction splitting schemes for the incompressible Navier-Stokes equations. The schemes combine a Legendre-spectral method for the spatial discretization and a pressure-stabilization/direction splitting scheme for the temporal discretization, leading to a sequence of one-dimensional elliptic equations at each time step while preserving the same order of accuracy as the usual pressure-stabilization schemes. We prove that these schemes are unconditionally stable, and present numerical results which demonstrate the stability, accuracy, and efficiency of the proposed methods.

Key words: Navier-Stokes equations, projection method, direction splitting, spectral methods.

1. Introduction

A main difficulty in solving the incompressible Navier-Stokes equations is how to deal with the divergence-free constraint which couples the velocity and the pressure. There exists an enormous amount of literature on this subject. A popular and efficient approach is to use a projection type method which was originated from the pioneering works of Chorin [2] and Temam [9]. This type of methods decouples the computation of pressure from that of velocity, and only requires to solve a sequence of Poisson type equations at each time step. We refer to [5] for an overview of the projection type methods.

Recently, Guermond and Minev [4] (see also [6]) proposed to combine the pressure-stabilization method (cf. [3, 10]) and the direction splitting technique [7] for the time discretization of the incompressible Navier-Stokes equations, leading to a sequence of one-dimensional problems at each time step. In [4] and [6], it is shown that these semi-discretized pressure-stabilization/direction splitting schemes are unconditionally stable and have the same order of accuracy as the usual projection schemes without spatial discretization.

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In this paper, we consider the stability of the spatial discretization of the pressure-stabilization/direction splitting schemes by using a Legendre-spectral method. It turns out that a usual Legendre-Galerkin or Legendre-collocation method will not lead to unconditionally stable schemes as in the semi-discretized case. To overcome this difficulty, we construct a hybrid of Legendre-Galerkin and Legendre-collocation methods which is easy to implement and enables us to prove the unconditional stability. We also construct a pressure-stabilization/direction splitting scheme for problems with variable viscosity and prove that a semi-discretized version is unconditionally stable.

The outline of the paper is as follows. In the next section we construct a fully discretized direction splitting scheme by using a hybrid of Legendre-collocation and Legendre-Galerkin methods. In Section 3, we carry out a stability analysis for the fully discretized schemes for both the standard and rotational versions. We present in Section 4 a generalization of the spectral direction splitting scheme to the Navier-Stokes equations with variable viscosity and prove a stability result in a simplified semi-discrete case. We present numerical results and discussions in Section 5.

2. Spectral Direction Splitting Schemes

The direction splitting schemes are usually applied to separable domains. We shall restrict our attention in this paper to $\Omega = (-1,1)^2$, and consider the time-dependent Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } \Omega\times(0,T], \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega\times[0,T], \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, & \text{in } [0,T], \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

where ν is the viscosity coefficient, $\boldsymbol{u}$ and p stand for the velocity vector and the pressure respectively. Since the treatment of the nonlinear term does not have an essential impact on the pressure-stabilization and direction splitting. We shall restrict our attention in this paper to the Stokes case, i.e., (2.1) without the nonlinear term.

2.1. Direction splitting scheme

We start with a second-order pressure-stabilization scheme:

$$\begin{cases} \dfrac{1}{\triangle t}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) - \nu\Delta\dfrac{\boldsymbol{u}^{n+1} + u^n}{2} + \nabla p^n = \boldsymbol{f}^{n+\frac{1}{2}}, \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$