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On Solution Regularity of Linear Hyperbolic Stochastic PDE Using the Method of Characteristics

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Abstract. The generalized Polynomial Chaos (gPC) method is one of the most widely used numerical methods for solving stochastic differential equations. Recently, attempts have been made to extend the the gPC to solve hyperbolic stochastic partial differential equations (SPDE). The convergence rate of the gPC depends on the regularity of the solution. It is shown that the characteristics technique can be used to derive general conditions for regularity of linear hyperbolic PDE, in a detailed case study of a linear wave equation with a random variable coefficient and random initial and boundary data.

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Key words: Hyperbolic equation, stochastic PDEs, regularity, characteristic method.

1. Introduction

Stochastic differential equations are of great importance, not only in a wide variety of scientific areas including non-equilibrium statistical physics and quantitative biology but also elsewhere such as in mathematical finance and economics. Since it is usually difficult to solve stochastic partial differential equations (SPDE) analytically, their numerical solution has generated considerable research interest in recent decades. A good introduction and review can be found in Ref. [2].

Among a number of proposed numerical methods, the generalized Polynomial Chaos (gPC) method is one of the most widely used. The basic idea of the gPC is to project the random part of the system into a finite dimensional polynomial space and then solve the projected finite dimensional deterministic system. Due to its spectral-method-like nature, one of the important properties of the gPC is that the rate of the convergence depends very much on the regularity of the solution. An excellent review of the gPC method can be found in Ref. [6]

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The gPC method has proven to be very efficient for various types of problems, but only recently have attempts been made to use it for hyperbolic SPDE [1,3,5]. While parabolic and elliptic equations generally have good regularity properties, hyperbolic problems have non-degenerate characteristics that can depend on the randomness, making the study of their regularity more difficult. This paper develops the characteristics technique to deal with the regularity of the solution to a particular linear SPDE, as a first step to tackle the regularity of hyperbolic SPDE in general.

The strategy of the characteristics technique is outlined in Section 2. In Section 3, the technique is used to derive a general condition for the BV regularity of the solution to the linear wave equation with a random variable coefficient and random initial and boundary data. Conclusions are made in Section 4.

2. Outline of the Characteristic Approach

An energy-estimate-type technique can be applied to linear hyperbolic SPDE to show BV regularity [5] . However, this technique requires some strong conditions. In the case of a wave equation with a constant random wave speed, the derivative of the wave speed $c_y(y)$ must be bounded. Since the method of characteristics generally gives sharp estimates for linear hyperbolic equations, we intend to use it to improve the aforementioned result and obtain a more general theorem.

The idea of the characteristics approach is straightforward. Thus we find an implicit representation of the solution in terms of the given data using the characteristics and then bound its TV norm. For a general hyperbolic PDE with a random parameter but no shock, the characteristics can be written as

$$L(x,t,y)=s$$

and initial data can be parametrized by a random curve

$$\Gamma:\gamma(s)=(f(s),g(s),h(s,y))=(x,t,u(y)),$$

where y is a random variable. The first equation contains the information about the influence of the randomness on the directions of the characteristic curves, and the second equation contains the randomness of the initial data. Then the solution can be written as

$$u(x,t,y)=h(L(x,t,y),y),$$

and we see immediately that

$$\left\|u_y\right\|_{L^1}=\int_{-1}^{1}\rho(y)\int_{-1}^{1}\left|\frac{\partial h}{\partial s}\frac{\partial L}{\partial y}+\frac{\partial h}{\partial y}\right|dx\,dy,$$

where the physical interpretation of each term is self-evident.

Although the idea of the technique is simple, as shown below it is nontrivial and needs careful analysis, even when it is applied to a relatively simple problem. The major obstacle is the interaction between the boundary and the characteristics, which makes the parametrization of the solution in terms of initial data difficult.