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A Time Splitting Space Spectral Element Method for the Cahn-Hilliard Equation

Lizhen Chen[1] and Chuanju Xu[2,*]

[1] *Beijing Computational Science Research Center, Beijing 100084, China.*
[2] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, China.*

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Abstract. We propose and analyse a class of fully discrete schemes for the Cahn-Hilliard equation with Neumann boundary conditions. The schemes combine large-time step splitting methods in time and spectral element methods in space. We are particularly interested in analysing a class of methods that split the original Cahn-Hilliard equation into lower order equations. These lower order equations are simpler and less computationally expensive to treat. For the first-order splitting scheme, the stability and convergence properties are investigated based on an energy method. It is proven that both semi-discrete and fully discrete solutions satisfy the energy dissipation and mass conservation properties hidden in the associated continuous problem. A rigorous error estimate, together with numerical confirmation, is provided. Although not yet rigorously proven, higher-order schemes are also constructed and tested by a series of numerical examples. Finally, the proposed schemes are applied to the phase field simulation in a complex domain, and some interesting simulation results are obtained.

AMS subject classifications: 65M06, 65M12, 65N30, 65N35, 76A05

Key words: Cahn-Hilliard, time splitting schemes, spectral methods, error analysis.

1. Introduction

The Cahn-Hilliard equation, originally introduced by Cahn and Hilliard to describe the phase separation and coarsening phenomena in a melted alloy [3], has now been used to model many moving interface problems from fluid dynamics to materials science via a phase-field approach — e.g. see Refs. [4, 5, 9, 21, 23, 25, 26, 28, 31]. Usually, the Cahn-Hilliard equation takes the form

$$\partial_t u + \Delta\left(\Delta u - \frac{1}{\varepsilon^2} f(u)\right) = 0\,, \quad 0 < t \le T, \boldsymbol{x} \in \Omega\,, \tag{1.1}$$

*Corresponding author. *Email addresses:* lzchen@csrc.ac.cn (L. Chen), cjxu@xmu.edu.cn (C. Xu)

where $\Omega \subset R^d$ (for $d = 2, 3$) is a bounded domain, $f(u) = F'(u)$ with $F(u)$ a given energy potential, the parameter ε denotes the interfacial width that is small compared to the characteristic length of the problem under consideration, and Δ denotes the Laplacian operator. This fourth-order equation can be viewed as the gradient flow of the Liapunov energy functional

$$E(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2} F(u) \right] d\boldsymbol{x} \tag{1.2}$$

in the space $H^{-1}(\Omega)$.

Another commonly used model in the investigation of moving surface problems is the Allen-Cahn equation (e.g. see Refs. [1, 10, 11, 19, 22]):

$$\partial_t u - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0 \,, \tag{1.3}$$

which was first introduced by Allen and Cahn [1] to describe the motion of antiphase boundaries in crystalline solids. Similar to the Cahn-Hilliard equation, Eq. (1.3) can be viewed as the gradient flow of the same Liapunov energy functional $E(u)$ defined in Eq. (1.2) in the space $L^2(\Omega)$. Both equations (1.1) and (1.3) satisfy the energy law

$$\partial_t E(u(t)) \leq 0 \,,$$

but each of them has its own advantages. Roughly speaking, the Allen-Cahn equation satisfies the maximum principle, and the computation is cheaper since it is a second-order equation. By contrast, the Cahn-Hilliard equation does not satisfy the maximum principle, but possesses conservation of total mass for the system, which is very important for some practical applications — e.g. the mixture of two incompressible fluids [23] or liquid crystal flows [24, 27].

Numerical methods for the Cahn-Hilliard equation can be found in many references — cf. [6–8, 12–18, 20, 30, 32, 33] and other references therein. Existing numerical techniques include finite element schemes [7, 13, 15, 16, 20], finite difference approaches [2, 17], and combined spectral and large-time stepping methods [33]. Compared to a standard conforming finite element method, which requires that the approximation space is a subspace of H^2, the methods based on a splitting technique [7, 18, 29, 32] only require C^0-continuity of the approximate solution and are therefore easier to implement. In this article, we analyse an approach for the Cahn-Hilliard equation using splitting schemes in time and spectral element methods in space. Our main purpose is to establish the stability and convergence properties of the proposed scheme, together with an extension to the spectral element method for the spatial discretisation. We are aware that similar analysis has been carried out in Refs. [18, 32], where an error estimate was obtained assuming boundedness of the discrete solution, but that assumption is only valid in the one-dimensional case for the standard energy potential. To overcome this difficulty, we consider here the two-dimensional Cahn-Hilliard equation corresponding to a truncated potential $F(u)$ with quadratic growth