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H^1-Stability and Convergence of the FE, FV and FD Methods for an Elliptic Equation

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Abstract. We obtain the coefficient matrices of the finite element (FE), finite volume (FV) and finite difference (FD) methods based on P_1-conforming elements on a quasi-uniform mesh, in order to approximately solve a boundary value problem involving the elliptic Poisson equation. The three methods are shown to possess the same H^1-stability and convergence. Some numerical tests are made, to compare the numerical results from the three methods and to review our theoretical results.

AMS subject classifications: 35Q30, 65N30

Key words: Finite element method, finite difference method, finite volume method, Poisson equation, stability and convergence.

1. Introduction

Elliptic equations form one of the most common classes of partial differential equations (PDE). Numerical schemes to obtain approximate solutions to elliptic equations are fundamentally different from those for parabolic and hyperbolic equations. Many practical problems involve elliptic equations — e.g. the well known cases of steady heat flow and the irrotational flow of an inviscid incompressible fluid.

Finite element (FE), finite volume (FV) and finite difference (FD) methods are three standard approaches to the discretisation of PDE that are often used for their approximate solution [1, 5, 12, 13, 17, 18]. The FV method may be regarded as a generalisation of the FD method [3, 7, 8], and may be applied to arbitrary domains without much difficulty.

The relationship between the FE and FV methods applied to the two-dimensional Poisson equation has been discussed by Vanselow [17], who also considered the FV method

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with Voronoi boxes for discretising elliptic boundary value problems — and showed that the matrix of the linear system of equations for the FV method is equivalent to the matrix for the FE method if and only if the Delaunay triangulation and P_1-conforming element are used. However, he did not prove any convergence results in comparing the FE solution and FV solutions under weaker assumptions. Mattiussi [18] applied aspects of algebraic topology to the analysis of the FV and FE methods, illustrating the similarity between the discretisation strategies adopted by the two methods via a geometric interpretation of the role played by weighting functions involved with the respective finite elements. Recently, Xu and Zou [16] presented some convergent properties of both linear and quadratic simplicial FV methods for elliptic equations. They established an inf-sup condition in a simple fashion for the linear FV method on domains of any dimension, and proved that the solution via a linear FV method is "super-close" to that from a relevant FE method — cf. also [3,4,11].

If the partition of a domain possesses certain geometrical properties, the linear convergence of the FE method with respect to a special energy norm follows when the solution of the problem involving the Poisson equation belongs to H^2 space. It is notable that the same convergence results can be obtained for the FD and FE methods based on P_1-conforming elements under weaker assumptions. In Section 2, we obtain the coefficient matrices of the FE, FD and FV methods based on continuous P_1-elements on a quasi-uniform grid to solve a boundary value problem involving the Poisson equation on a one-dimensional domain, and show that the three methods possess the same H^1-stability and convergence. The coefficient matrices of the three methods are provided and their H^1-stability and convergence are then analysed in a two-dimensional domain in Section 3. Numerical experiments are presented in Section 4, to confirm the theoretical analysis and demonstrate the numerical results of the three methods.

2. FE, FV and FD Methods in One Dimension ($d = 1$)

In this article, we consider applying FE, FD and FV methods on a quasi-uniform grid for the boundary value problem involving the Poisson equation

$$-\Delta u = f \qquad (x_1, \cdots, x_d) \in \Omega \tag{2.1}$$

$$u = 0 \qquad (x_1, \cdots, x_d) \in \partial\Omega \tag{2.2}$$

on a bounded domain $\Omega \subset R^d$ with boundary $\partial\Omega$, where $\Delta = \partial_{x_1 x_1} + \cdots + \partial_{x_d x_d}$ is the Laplacian corresponding to the gradient operator $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})^\top$. We always assume $f \in L^2(\Omega)$, and the regularity estimate

$$\|u\|_{2,\Omega} \le c\|f\|_{0,\Omega} . \tag{2.3}$$

In this section, we consider the one-dimensional case $d = 1$, with the assumed domain $\Omega = (0,1)$ sub-divided as $0 = x_0 < x_1 < \cdots < x_{m+1} = 1$ — i.e. into sub-intervals $I_i = x_{i-1}, x_i)$ of size $h_i = x_i - x_{i-1}$ (cf. Fig. 1). Let us also define $x_{i+\frac{1}{2}} = (x_i + x_{i+1})/2$ for $i = 0, \cdots, m$, and set $h = \max_{1 \le i \le m+1} h_i$.