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Computing Switching Surfaces in Optimal Control Based on Triangular Decomposition

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Abstract. Various algorithms for optimal control require the explicit determination of switching surfaces. However, switching strategies may be very complicated, such that the computation of switching surfaces is quite challenging. General methods are proposed here to compute switching surfaces systematically, based on algebraic computational tools such as triangular decomposition. Our methods are highly complex compared to some widely-used numerical options, but they can be made feasible for real-time applications by moving the computational burden off-line. The tutorial-style presentation is intended to introduce potentially powerful symbolic computation methods to system scientists in particular, and an illustrative example of time-optimal control is given to show the effectiveness and generality of our approach.

AMS subject classifications: 13P15, 13P25

Key words: Optimal control, switching surface, triangular decomposition, semi-algebraic system.

1. Introduction

Optimal control has been used in many areas of modern system science such as aeronautics, astronautics, robotics and power electronics [8, 13, 16, 26, 41]. The control algorithms typically require explicit determination of switching surfaces — surfaces where the sign of the control input changes. However, switching strategies may be very complicated in many practical applications, so the computation of switching surfaces becomes quite challenging. Some special approaches to their computation have been developed [1, 18, 24]. Walther *et*

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al. [30] introduced tools from computational algebraic geometry for time-optimal control problems, by transforming the computation of switching surfaces into a combinatorial problem. Their main idea is to first compute Gröbner bases of particular polynomial equations deduced from the original system, and then use Sturm's theorem to determine whether the equations have non-negative solutions. Their approach is somewhat non-systematic and may not be feasible for real-time control due to its high complexity, which motivated the development of more general systematic methods for computing switching surfaces presented here. Another purpose of this article is to introduce powerful methods of symbolic computation (coupled with numerical computation) for system scientists.

Our methods are based on triangular decomposition and related algebraic tools. Like Gröbner bases, triangular decomposition is a main elimination approach for solving systems of multivariate polynomial equations. For example, consider the equations

$$\begin{cases} P_1 = x_1^2 + x_2 + x_3 - 1 = 0\,, \\ P_2 = x_2^2 + x_3 + x_1 - 1 = 0\,, \\ P_3 = x_3^2 + x_1 + x_2 - 1 = 0\,. \end{cases}$$

Under the variable ordering $x_1 < x_2 < x_3$, triangular decomposition of the polynomial set $\mathscr{P} = \{P_1, P_2, P_3\}$ results in

$$\mathscr{T}_1 = [x_1^2 + 2x_1 - 1, x_2 - x_1, x_3 - x_1], \quad \mathscr{T}_2 = [x_1 - 1, x_2, x_3]\,,$$
$$\mathscr{T}_3 = [x_1, x_2 - 1, x_3], \quad \mathscr{T}_4 = [x_1, x_2, x_3 - 1]\,,$$

such that the union of the zero sets of $\mathscr{T}_1, \cdots, \mathscr{T}_4$ is identical to the zero set of $\mathscr{P}$. The *triangular sets* $\mathscr{T}_1, \cdots, \mathscr{T}_4$ are of triangular form. Note that the zero set of a triangular set can readily be obtained by successively computing the zeros of its polynomials. In our algorithms, triangular decomposition is thus used as a preprocessing tool in the analysis of solutions of polynomial equations. Formal notation and properties related to triangular decomposition are provided in subsection 3.1 below.

Since triangular decomposition is such a key aspect of our approach, a brief literature overview may be helpful. In considering differential ideals, Ritt [25] introduced the notion of characteristic set, one of the best known concepts for triangular sets. Several decades later, Wu [37, 38] extended Ritt's work by removing irreducibility requirements of characteristic sets, proposing efficient algorithms for decomposing polynomial sets, and successfully applying them to geometric theorem proof. Wu's method was intensively studied and improved by a number of researchers (e.g. [4, 9–11, 21, 34, 35], but the zero set of a characteristic set may be empty. To avoid this degeneracy, Kalkbrener [15] and Yang *et al.* [39] independently introduced the notion of a regular set, and proposed methods for decomposing any algebraic variety into finite components represented by regular sets. Wang [31] proposed another method for triangular decomposition, which is considered to be quite efficient. Other relevant work on the triangular decomposition of polynomial sets is discussed in Refs. [3, 17, 23, 33] (and references therein).

Our work presented here is based on the observation that the problem of computing the switching surfaces of the time-optimal control can be translated into identifying whether