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# Momentum Conservative Schemes for Shallow Water Flows

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**Abstract.** We discuss the implementation of the finite volume method on a staggered grid to solve the full shallow water equations with a conservative approximation for the advection term. Stelling & Duinmeijer [15] noted that the advection approximation may be energy-head or momentum conservative, and if suitable which of these to implement depends upon the particular flow being considered. The momentum conservative scheme pursued here is shown to be suitable for 1D problems such as transcritical flow with a shock and dam break over a rectangular bed, and we also found that our simulation of dam break over a dry sloping bed is in good agreement with the exact solution. Further, the results obtained using the generalised momentum conservative approximation for 2D shallow water equations to simulate wave run up on a conical island are in good agreement with benchmark experimental data.

**AMS subject classifications**: 65M08, 76M12, 35L65

**Key words**: Finite volume method, staggered grid, conservative scheme, shallow water equations.

## 1. Introduction

Shallow water equations (SWE) are often used to model fluid flow in rivers, lakes, estuaries or coastal areas. The main assumption in adopting a SWE model is that the horizontal length scale is much greater than the vertical scale — and for example, complex changes to rapidly varying flows in coastal hydrodynamics may be described, including the inundation of dry land. From the mathematical point of view, the mass and momentum conservation equations involved constitute an hyperbolic system, and the finite volume method is known to be very effective for computations involving equations derived from conservation laws (such as in SWE modelling).

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Unlike finite difference methods, where derivatives at a point are replaced with truncated Taylor series, in finite volume methods the focus is on approximating the conservative equations in discrete form on a fixed volume in space called a cell. The flux entering a given cell is identified with that leaving an adjacent cell or cells, so finite volume methods are known as conservative schemes. In Godunov, Suliciu or other relaxation schemes, the two equations in the SWE are approximated on the same cell and the numerical fluxes are computed using an approximate Riemann solver — cf. [2, 10, 14, 17, 18] for descriptions and analysis of this approach. Refs. [8, 13, 19] present thorough investigations of a high-order finite volume scheme applied to uniform or non-uniform meshes for the SWE model. Alternatively, finite volume schemes can involve approximations on adjacent cells, resulting in schemes where the two unknowns of the SWE are calculated on a staggered grid. Implementing a finite volume scheme on a staggered grid has a great advantage, for the numerical fluxes can be computed more simply since the need to use an approximate Riemann solver can be avoided. Staggered finite volume approximations for solving nonlinear hyperbolic systems involving conservation laws have been investigated in the past few years [4, 9, 15, 16]. The advection term in a SWE model is the most difficult to approximate, and Stelling & Duinmeijer [15] discuss two alternative approaches (energy-head or momentum conservative) — although their applicability depends upon the particular problem. The appropriate choice of conservative method for topographies with abrupt changes has been discussed [15],

In this article, we show that the momentum conservative approach is suitable for dam break simulation for various bottom topographies — and also for simulations of 2D wave run up on a conical island, found to be in good agreement with experimental data [3] previously used for validating various nonlinear shallow water equations solvers [6,12,20]. Our discussion starts in Section 2 with a description of the finite volume method on a staggered grid for a simple linear SWE model. The momentum conservative scheme for a nonlinear 1D SWE model is then discussed in Section 3, and in Section 4 the scheme is implemented for various shallow water flows. The analogous conservative scheme for 2D shallow water equations is described in Section 5, and in Section 6 we present the simulation of run up waves on a conical island and compare the experimental data of Ref. [3].

## 2. Finite Volume Method on a Staggered Grid for a Simple Linear SWE Model

We first discuss the formulation of a leapfrog method for a linear SWE model, and stress its equivalence with the finite volume method on a staggered grid. This provides a solid building block for further development since the method is explicit, non-dissipative, and conditionally stable.

The simple governing equations for small amplitude gravity waves above a flat bottom $d_0$ are

$$\eta_t + d_0 u_x = 0\,, \tag{2.1}$$

$$u_t + g\eta_x = 0\,, \tag{2.2}$$