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Three Iterative Finite Element Methods for the Stationary Smagorinsky Model

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Abstract. Three iterative stabilised finite element methods based on local Gauss integration are proposed in order to solve the steady two-dimensional Smagorinsky model numerically. The Stokes iterative scheme, the Newton iterative scheme and the Oseen iterative scheme are adopted successively to deal with the nonlinear terms involved. Numerical experiments are carried out to demonstrate their effectiveness. Furthermore, the effect of the parameters *Re* (the Reynolds number) and δ (the spatial filter radius) on the performance of the iterative numerical results is discussed.

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Key words: Smagorinsky model, stabilised finite element method, local Gauss integration, iterative scheme, lid driven cavity flow.

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1. Introduction

Large eddy simulation (LES) has attracted much attention over the last two decades, especially because increased computational resources have extended the range of scales that LES models might simulate. The LES approach is based upon a simple computational idea — i.e. approximate only the large structures in the flow, while modelling the influence of smaller ones. The large structures are defined by convolving the flow variables with a spatial filter of radius δ. To model the effect of the discarded small structures, traditionally physical insight from the statistical theory of turbulence (such as the energy cascade) has been used. The Smagorinsky model is one of the most popular LES models, where this physically-based approach involves introducing an artificial viscosity term that dissipates energy in the large scale structure at the same rate as the discarded small structures would have dissipated had they been included in the model.

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In this article, we consider the following steady Smagorinsky model:

$$-\nu\Delta u - \nabla\cdot((C_S\delta)^2|\nabla u|\nabla u) + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega\,, \tag{1.1}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega\,, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega\,. \tag{1.3}$$

Here Ω is a bounded, convex and open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\Omega$, u represents the velocity vector, p the pressure, f the prescribed spatially filtered forcing term, C_S the Smagorinsky constant, δ the radius of the spatial filter used in the LES model, $\nu > 0$ the viscosity inversely proportional to the Reynolds number Re, and $|\sigma| = \sqrt{\sum_{i,j=1}^{2}|\sigma_{ij}|^2}$ the Frobenius norm of the tensor σ.

There are numerous works devoted to the development of efficient schemes for solving the stationary Smagorinsky model [5–8, 13]. It is well known that numerical computations for the stationary Smagorinsky model can be performed by iterative procedures, and in practice one usually takes the solution of the Stokes equations as the initial iterative input. The stationary Smagorinsky equations are then solved by an iterative procedure until the norm of the difference in successive iterations falls within a fixed tolerance. Usually, efficient approximations of the transient Smagorinsky equations are based on a semi-discretisation in time, followed by a spatial discretisation at each time step — e.g. using finite elements to solve the stationary Smagorinsky equations at each time step. However, a search for the most efficient numerical methods to solve the stationary Smagorinsky model is justified.

Finite element methods (FEMs) are widely used in computational fluid dynamics. In particular, some stable mixed FEMs are often a basic component in efficiently solving the incompressible flow equations. Of the mixed element methods, equal-order velocity-pressure pairs have proven quite practical in finite element approximations of the Smagorinsky problem, but they violate the inf-sup condition [14] and the compatibility between the velocity and pressure spaces. In using a primitive variable formulation, the importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf-sup condition is therefore widely understood. This condition has played an important role because it ensures a stability and accuracy of the underlying numerical schemes — thus a pair of finite element spaces to approximate the velocity and pressure unknowns are said to be stable if they satisfy the inf-sup condition. Intuitively, this condition enforces a certain correlation between the two finite element spaces, so that they both have the required properties when employed for the Navier-Stokes equations.

However, due to computational convenience and efficiency in practice, some mixed finite element pairs that do not satisfy the inf-sup condition are also popular. Consequently, considerable attention has been paid to the study of stabilised methods for the Stokes and Navier-Stokes problems. Recent studies have focused on stabilisation of the lowest equal-order finite element pair (piecewise linear polynomials) using the projection of the pressure onto the piecewise constant space [2, 4, 5, 9, 10, 12], a technique that does not require specification of stabilising parameters or edge-based data structure. There are some important advantages over traditionally stabilised mixed finite element methods —