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A Third-Order Accurate Direct Eulerian GRP Scheme for One-Dimensional Relativistic Hydrodynamics

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Abstract. A third-order accurate direct Eulerian generalised Riemann problem (GRP) scheme is derived for the one-dimensional special relativistic hydrodynamical equations. In our GRP scheme, the higher-order WENO initial reconstruction is employed, and the local GRPs in the Eulerian formulation are directly and analytically resolved to third-order accuracy via the Riemann invariants and Rankine-Hugoniot jump conditions, to get the approximate states in numerical fluxes. Unlike a previous second-order accurate GRP scheme, for the non-sonic case the limiting values of the second-order time derivatives of the fluid variables at the singular point are also needed for the calculation of the approximate states; while for the sonic case, special attention is paid because the calculation of the second-order time derivatives at the sonic point is difficult. Several numerical examples are given to demonstrate the accuracy and effectiveness of our GRP scheme.

AMS subject classifications: 65M06, 76M12, 76Y05

Key words: Godunov-type scheme, WENO, generalised Riemann problem, Riemann invariant, Rankine-Hugoniot jump condition, relativistic hydrodynamics.

1. Introduction

If the fluid velocity is locally close to light speed in a vacuum or the internal energy density is locally comparable (or larger) than the fluid rest-mass density, a relativistic description of the fluid dynamics should be adopted. Moreover, the Einstein field theory of gravity is appropriate whenever the matter is influenced by large gravitational potentials. Relativistic flows arise in numerous astrophysical phenomena, from stellar to galactic scales — e.g. active galactic nuclei, super-luminal jets, core collapse super-novae, pulsars, coalescing neutron stars, black holes, micro-quasars, X-ray binaries, and gamma-ray bursts.

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Theoretical relativistic flow dynamics involves solving highly nonlinear equations, where an analytic treatment is extremely difficult so that numerical studies are important. The earliest numerical study can be traced to Ref. [16], where the general relativistic hydrodynamic (RHD) equations in Eulerian form are solved by an explicit finite difference method using an artificial viscosity. Subsequently, other relevant finite difference methods were systematically introduced [17]. Various modern shock-capturing methods based on exact or approximate Riemann solvers have since been developed for the RHD equations — cf. the review articles [5,13], and more recent overviews of numerical methods for the RHD equations [18,24]. Second-order accurate direct Eulerian generalised Riemann problem (GRP) schemes have recently been proposed for both 1D and 2D relativistic hydrodynamics [21,22]. An analytic extension of the Godunov method, the GRP scheme was originally devised for non-relativistic compressible fluid dynamics [1], by utilising a piecewise linear function to approximate the initial data and then analytically resolving a local GRP at each interface to yield numerical fluxes — cf. the comprehensive description in Ref. [2] and references therein.

The original GRP scheme has two versions, Lagrangian and the Eulerian. The Eulerian form is always derived using the Lagrangian framework, which has the advantage that the contact discontinuity in each local wave pattern is always fixed with speed zero and the nonlinear waves are located on either side. However, the passage from the Lagrangian framework to the Eulerian form is sometimes quite delicate, particularly for the sonic case and multi-dimensional applications. To avoid the difficulty, second-order accurate direct Eulerian GRP schemes were respectively developed for the shallow water equations [8], the Euler equations [4], the governing equations for the gas-liquid two-phase flow in HTHP transient wells [20], and a more general weakly coupled system [3] by directly and analytically resolving the local GRPs in the Eulerian formulation via Riemann invariants and the Rankine-Hugoniot jump conditions. The GRP scheme has been compared with the gas-kinetic scheme for inviscid compressible flow simulations [9]. In Ref. [6], the adaptive direct Eulerian GRP scheme was further developed with improved resolution as well as accuracy by combining the moving mesh method [15]; and the accuracy and performance of the adaptive GRP scheme was further studied in simulating 2D complex wave configurations formulated via 2D Riemann problems from compressible Euler equations [7].

The aim of this article is to derive a third-order accurate direct Eulerian GRP scheme for the 1D special RHD equations analytically, extending the recent second-order accurate direct Eulerian GRP scheme for the RHD equations [21,22] and the third-order accurate direct Eulerian GRP scheme for 1D and 2D non-relativistic Euler equations [19]. In passing, we note that a unified approach for solving the GRP with higher-order accuracy has been provided for general 1D hyperbolic balance laws [14]. Section 2 introduces the 1D special RHD equations and corresponding Riemann invariants as well as their basic properties. The third-order accurate GRP scheme for the 1D RHD equations is derived analytically in Section 3. The scheme is first outlined in Section 3.1, and the local GRPs are resolved in Section 3.2 — with the rarefaction and shock waves respectively discussed in Subsections 3.2.1 and 3.2.2, the approximate states in numerical fluxes separately for both non-sonic and sonic cases in Subsection 3.2.3, and the acoustic case in Subsection 3.2.3.