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An Algorithm for the Proximity Operator in Hybrid TV-Wavelet Regularization, with Application to MR Image Reconstruction

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Abstract. Total variation (TV) and wavelet L_1 norms have often been used as regularization terms in image restoration and reconstruction problems. However, TV regularization can introduce staircase effects and wavelet regularization some ringing artifacts, but hybrid TV and wavelet regularization can reduce or remove these drawbacks in the reconstructed images. We need to compute the proximal operator of hybrid regularizations, which is generally a sub-problem in the optimization procedure. Both TV and wavelet L_1 regularisers are nonlinear and non-smooth, causing numerical difficulty. We propose a dual iterative approach to solve the minimization problem for hybrid regularizations and also prove the convergence of our proposed method, which we then apply to the problem of MR image reconstruction from highly random under-sampled k-space data. Numerical results show the efficiency and effectiveness of this proposed method.

AMS subject classifications: 65K10, 68U10

Key words: Total Variation (TV), wavelet, regularization, MR image.

1. Introduction

In many image restoration or reconstruction problems, we need to solve a linear inverse problem of the form

$$\boldsymbol{g} = K\boldsymbol{f} + \boldsymbol{n}\,,$$

where $\boldsymbol{g}$ is the observed data, K is the system operator, $\boldsymbol{f}$ is the original image with size $m \times n$ and $\boldsymbol{n}$ is the random noise. It is well known that restoring an image is a very ill-conditioned process, and to alleviate this a regularization approach is generally used. The approach is to minimise the objective function, which is the weighted sum of the data-fitting term and the term containing some prior information about the original image.

In many image processing problems, an image can be modelled as a piecewise smooth function, and simultaneously sparsely represented by a wavelet basis — e.g. Lustig *et*

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al [13] illustrate such sparsity in the transform domain of MR images and piecewise smoothness in the spatial domain of angiogram images. The images consequently have both small total variation (TV) norm [16] and small L_1 norm, and the reconstructed image $\boldsymbol{f}$ is a minimizer of the objective function

$$\min_{\boldsymbol{f}} D(\boldsymbol{g},\boldsymbol{f}) + \lambda_1 \mathrm{TV}(\boldsymbol{f}) + \lambda_2 \left\| \boldsymbol{W}\boldsymbol{f} \right\|_1 , \tag{1.1}$$

where $D(\boldsymbol{g},\boldsymbol{f})$ is the data-fitting term that denotes a discrepancy measure between the observed data $\boldsymbol{g}$ and the solution $\boldsymbol{f}$, and $\lambda_i (i = 1,2)$ is the regularization parameter. The term $\mathrm{TV}(\boldsymbol{f})$ denotes the TV norm of the image $\boldsymbol{f}$, which can preserve edges in the image due to the piecewise smooth regularization property of the TV norm, but it may over-smooth image details and introduce staircase effects. While wavelet L_1 regularization can keep local image features and details through sparse representation of the image, it may introduce some ringing artifacts along image contours. The main advantage in combining TV regularization with the L_1 norm of wavelet coefficients is to reduce or remove staircase effects caused by TV regularization and ring effects caused by wavelet regularization.

The chief challenge in solving the problem (1.1) is that the TV and L_1 regularisers are both nonlinear and non-smooth. The minimizer of (1.1) can be computed by the conjugate gradient method [13] or PDE approach [12], but the main drawback is that the convergence is very slow in practice. When the data fitting term $D(\boldsymbol{g},\boldsymbol{f})$ has a Lipschitz-continuous gradient, it is possible to use the forward-backward splitting proximal algorithm to solve the optimization problem [8]. The proximity operator of the function $\psi(\boldsymbol{f})$ is defined as

$$\mathrm{prox}_{\psi}(\boldsymbol{u}) = \arg\min_{\boldsymbol{f}} \frac{1}{2} \left\| \boldsymbol{f} - \boldsymbol{u} \right\|_2^2 + \psi(\boldsymbol{f}) , \tag{1.2}$$

where $\psi(\boldsymbol{f}) = \lambda_1 \mathrm{TV}(\boldsymbol{f}) + \lambda_2 \left\| \boldsymbol{W}\boldsymbol{f} \right\|_1$. Applying forward-backward splitting proximal algorithm, the solution of the problem (1.1) is given by

$$\boldsymbol{f} = \mathrm{prox}_{\alpha\psi} \left(\boldsymbol{f} - \alpha \nabla_{\boldsymbol{f}} D(\boldsymbol{g},\boldsymbol{f}) \right)$$

where $\alpha > 0$, which suggests that the minimizer $\boldsymbol{f}$ might be achieved by performing an iterative scheme with an initial solution.

However, an important task in forward-backward splitting proximal algorithm is to compute the proximal operator of the regularisers. Chambolle [3] proposed a project algorithm to compute the proximal operator of a TV regulariser, and it is well known that the proximal operator of a wavelet L_1 regulariser is a shrinkage operator [9]. Combettes & Pesquet developed an iterative method to compute the proximity operator of composite regularisers, by performing the proximity operator of each regulariser independently [7]. Recently, we obtained a formulation to compute the proximal operator when the function ψ is a linear combination of a TV norm and wavelet L_1 norm [2], but the relevant convergence analysis was not given there. In this article, we reconsider how to compute the proximal operator of the linear combination of the TV and wavelet L_1 norms — i.e. we study the minimization problem

$$\min_{\boldsymbol{f}} \mathbf{Q}(\boldsymbol{f}) \equiv \frac{1}{2} \left\| \boldsymbol{f} - \boldsymbol{g} \right\|_2^2 + \lambda_1 \mathrm{TV}(\boldsymbol{f}) + \lambda_2 \left\| \boldsymbol{W}\boldsymbol{f} \right\|_1 . \tag{1.3}$$