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Tikhonov Regularisation Method for Simultaneous Inversion of the Source Term and Initial Data in a Time-Fractional Diffusion Equation

Zhousheng Ruan[1,2], Jerry Zhijian Yang[1,3,*] and Xiliang Lu[1,3]

[1] *School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P.R. China.*
[2] *School of Science, East China Institute of Technology, Nanchang, Jiangxi, 330013, P.R. China.*
[3] *Computational Science Hubei Key Laboratory, Wuhan University, Wuhan, 430072, P.R. China.*

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Abstract. The inverse problem of identifying the time-independent source term and initial value simultaneously for a time-fractional diffusion equation is investigated. This inverse problem is reformulated into an operator equation based on the Fourier method. Under a certain smoothness assumption, conditional stability is established. A standard Tikhonov regularisation method is proposed to solve the inverse problem. Furthermore, the convergence rate is given for an *a priori* and *a posteriori* regularisation parameter choice rule, respectively. Several numerical examples, including one-dimensional and two-dimensional cases, show the efficiency of our proposed method.

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Key words: Time-fractional diffusion equation, conditional stability, Tikhonov regularisation, Morozov discrepancy principle, convergence rate.

1. Introduction

Let Ω denote an open bounded domain in $\mathbb{R}^d$ $(d = 1,2,3)$ with sufficiently smooth boundary $\partial\Omega$, and let us consider the following time-fractional diffusion equation with homogeneous Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + Lu(x,t) = f(x)\,, & (x,t) \in \Omega \times (0,T)\,, \\ u(x,t)|_{\partial\Omega} = 0\,, & (x,t) \in \partial\Omega \times (0,T]\,, \\ u(x,0) = u^0(x)\,, & x \in \Omega\,, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `zhshruan@whu.edu.cn` (Z. Ruan), `zjyang.math@whu.edu.cn` (J. Z. Yang), `xllv.math@whu.edu.cn` (X. Lu)

where $\alpha \in (0,1)$ and L is a symmetric strongly elliptic operator given by

$$L(u) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} \theta_{i,j} \frac{\partial}{\partial x_j} u(x)\right) + c(x)u(x)\,.$$

Assume that

$$\theta_{i,j} \in C^1(\overline{\Omega}),\ \theta_{i,j} = \theta_{j,i}, c(x) \in C(\overline{\Omega}),\ c(x) \geq 0, \forall x \in \overline{\Omega}\,,$$

and there exists a constant $\nu > 0$ such that $\nu \sum_i^d \xi_i^2 \leq \sum_{i,j=1}^d \theta_{i,j}(x)\xi_i\xi_j\ \forall x \in \overline{\Omega},\ \xi \in \mathbb{R}^d$. The fractional derivative $\partial^\alpha u(x,t)/\partial t^\alpha$ is the Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial u}{\partial \eta} \mathrm{d}\eta,\ 0 < \alpha < 1\,, \tag{1.2}$$

where $\Gamma(1-\alpha)$ is the Gamma function.

The time-fractional diffusion equation (1.1) has received much attention recently, due to many applications in various areas of engineering. The mathematical theory and the associated numerical method for the anomalous diffusion equation have often been discussed — e.g. see [1, 5, 9, 10, 13–15] and references therein. The inverse problem for the time-fractional diffusion equation has also been studied extensively. For example, the backward problem was explored by the quasi-reversibility method, the optimisation method, the data regularisation method, and the spectral truncation method [11, 17, 18, 22, 28]. In Refs. [26, 29], the eigenfunction expansion and integral equation methods were applied to recover the space-dependent or time-dependent source term, respectively. The inverse boundary problem, inverse potential problem, inverse coefficient problem and the order identification problem have all been investigated [6, 8, 12, 27].

Here we consider the reconstruction of the source term $f(x)$ and initial condition $u(x,0) = u^0(x)$ from the noisy measurement $g_i^\delta \approx u(x,T_i),\ i = 1,2$ with $0 < T_1 < T_2$ and the noise level such that

$$\left\|u(x,T_1) - g_1^\delta\right\| \leq \delta\,, \qquad \left\|u(x,T_2) - g_2^\delta\right\| \leq \delta\,. \tag{1.3}$$

When $\alpha = 1$, this is the simultaneous identification problem for the diffusion equation, which has been studied using various approaches [7, 24, 25, 30].

Some work has been done on the convergence analysis for Tikhonov regularisation of the backward problem or the source inverse problem. For example, in Ref. [20] the Tikhonov regularisation method was provided to solve a backward problem for a time-fractional diffusion equation, and the respective convergence rates were obtained for an *a priori* or an *a posteriori* regularisation parameter choice rule. A space-dependent source identification problem and corresponding convergence estimates for the Tikhonov regularisation method have been considered [21]. However, to the best of our knowledge the problem of simultaneously identifying the initial data and the source term for the time-fractional diffusion equation has not been solved previously.

Some preliminary results are given in Section 2. In Section 3, the ill-posedness of the inverse problem and a conditional stability result are provided. In Section 4, the Tikhonov