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An AMG Preconditioner for Solving the Navier-Stokes Equations with a Moving Mesh Finite Element Method

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Abstract. AMG preconditioners are typically designed for partial differential equation solvers and divergence-interpolation in a moving mesh strategy. Here we introduce an AMG preconditioner to solve the unsteady Navier-Stokes equations by a moving mesh finite element method. A $4P1-P1$ element pair is selected based on the data structure of the hierarchy geometry tree and two-layer nested meshes in the velocity and pressure. Numerical experiments show the efficiency of our approach.

AMS subject classifications: 65M60

Key words: Navier-Stokes system, algebraic multigrid precondition, moving mesh.

1. Introduction

We consider the incompressible Navier-Stokes equations in primitive variables

$$\begin{aligned} &\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p - \nu \nabla^2 \boldsymbol{u} = \boldsymbol{f}\,, \\ &\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,, \end{aligned} \tag{1.1}$$

subject to initial and boundary conditions on $\partial\Omega = \partial\Omega_D \bigcup \partial\Omega_N$ — viz.

$$\begin{aligned} &\boldsymbol{u} = \mathbf{w} \quad \text{on } \partial\Omega_D \times [0,T]\,, \\ &\nu \partial_{\boldsymbol{n}} \boldsymbol{u} - \boldsymbol{n} p = \mathbf{0} \quad \text{on } \partial\Omega_N \times [0,T]\,, \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \quad \text{in } \Omega\,, \end{aligned} \tag{1.2}$$

where $\Omega \in \mathscr{R}^d$ $(d = 2,3)$ is the spatial domain, $[0,T]$ the time interval, $\boldsymbol{u}$ the velocity vector and p the scalar pressure, $\boldsymbol{n}$ denotes the outward normal to the boundary $\partial\Omega$ of Ω, $\nu > 0$ the constant kinematic viscosity coefficient, and $\boldsymbol{f}$ represents the total external force. In particular, here we discuss the solution of Eqs. (1.1) and (1.2) by moving mesh finite element methods.

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After Winslow [1] proposed using a moving mesh to solve elliptic partial differential equations, Dvinsky [2] pointed out harmonic function theory could be used to generate a mesh, which motivated Li *et al.* [3] to propose a moving mesh finite element strategy based upon harmonic mapping. Di *et al.* [4] extended the moving strategy to solve the incompressible Navier-Stokes equations in primitive variables. The present authors [5] then designed a divergence-free interpolation in a moving strategy for solving a system of linearised Navier-Stokes-type equations and the incompressible Navier-Stokes flow equations, by applying a $4P1-P1$ element pair with a moving mesh finite element method. This pair has the same mesh structure as a $P1isoP2P1$ element, which is naturally LBB stable [6]. Four velocity elements can be obtained by refining the pressure element once — cf. Fig. 1. In the $4P1-P1$ element, both velocity and pressure are all standard $P1$, but in the $P1isoP2P1$ element they are not — see Ref. [5] for details.

Spatial discretisation of the Navier-Stokes system via the LBB-stable $4P1-P1$ element pair leads to a saddle point problem. A two-grid method was introduced to solve the Navier-Stokes equations [7–9]. There are many articles on saddle point problems where preconditioners for the Krylov subspace method are developed, such as in block and multigrid preconditioning [10]. Many authors have introduced various block preconditioners, where the main issue is to find a good approximation of the Schur complement [11–14]. There are other preconditioned methods discussed in Refs. [15–17], involving an efficient AMG preconditioner for a Krylov solver for the Navier-Stokes equations. Efficient precondition methods for saddle point problems are mainly based on a uniform mesh, although a stretched mesh was considered in Ref. [16].

Here we apply an AMG preconditioner to a finite element moving mesh method for solving the Eqs. (1.1) and (1.2), based on the work of Ref. [18]. An AMG precondition stretegy is designed for divergence-free interpolation in our moving mesh method, and the efficiency of our AMG preconditioner is analysed through several numerical experiments. In Section 2, we use the $4P1-P1$ element to approximate the governing equations, and present our AMG preconditioner for the Navier-Stokes equations in Section 3. In Section 4, we briefly discuss our moving mesh strategy, and then present numerical experiments in Section 5. Our concluding remarks are in Section 6.

2. Data Structure and Weak Formulation

We divide the time interval $[0,T]$ into N steps with $\{t_i\}_{i=1}^N$, and let $\mathbf{u}^j$ and p^j denote the discrete approximation to $\boldsymbol{u}(\cdot,t_j)$ and $p(\cdot,t_j)$. For simplicity, we choose the backward Euler scheme in linearising the term $\boldsymbol{u}^{n+1}\cdot\nabla\boldsymbol{u}^{n+1}$ as $\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^{n+1}$. We adopt the finite element pair $4P1-P1$, in choosing two different triangular meshes and two different finite element spaces. By using the hierarchy geometry tree structure [19], the velocity mesh involving a global refining pressure mesh needs to be generated only once — cf. Fig. 2.

The $1-1$ index between the velocity elements and pressure elements can be obtained readily via the hierarchy geometry tree structure — cf. Ref. [5] for details. Let $\mathscr{T}_h$ denote the triangular subdivision for the velocity mesh with mesh size $h=\max_{T\in\mathscr{T}_h} diam(T)$, $\mathscr{T}_H(H=2h)$ the pressure mesh, and $X^h\subset(H_0^1(\Omega)^2)$ and $P^H\subset L^2(\Omega)$ the respective