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A Block Diagonal Preconditioner for Generalised Saddle Point Problems

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Abstract. A lopsided alternating direction iteration (LADI) method and an induced block diagonal preconditioner for solving block two-by-two generalised saddle point problems are presented. The convergence of the LADI method is analysed, and the block diagonal preconditioner can accelerate the convergence rates of Krylov subspace iteration methods such as GMRES. Our new preconditioned method only requires a solver for two linear equation sub-systems with symmetric and positive definite coefficient matrices. Numerical experiments show that the GMRES with the new preconditioner is quite effective.

AMS subject classifications: 15A06, 65F10, 65H10

Key words: Generalised saddle point problem, Krylov subspace methods, alternating direction iteration, preconditioning, convergence.

1. Introduction

Recently, considerable attention has been devoted to solving large linear systems in saddle point form, which arise in a wide variety of scientific computing and engineering applications — e.g. computational fluid dynamics, mixed finite element approximation of elliptic partial differential equations, constrained least-squares problems, and structure analyses [1, 2, 6, 7, 14, 15, 19, 23, 25, 31, 33, 34]. In this article, we consider an iteration solution of the generalised saddle point linear system

$$\mathscr{A}x = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} = b\,, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite (SPD), $B \in \mathbb{R}^{n\times m}$ is of full column rank, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semi-definite (SPSD), $y, f \in \mathbb{R}^n$ and $z, g \in \mathbb{R}^m$ $(n \geq m)$.

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It is well known that iteration methods are preferable for large problems with sparse large coefficient matrices, and various methods for solving the linear systems such as (1.1) have been developed — e.g. Uzawa-type schemes [12,13,18,22,27], Hermitian and skew-Hermitian Splitting (HSS) method and its variants [5–11,21,35,37], and Krylov subspace methods [2, 4, 26, 30, 32]. However, Krylov methods for example can be very slow or even fail to converge if not conveniently preconditioned, and an acceptable preconditioner should reduce the number of iterations required for convergence but not significantly increase the amount of computation required at each iteration. Many preconditioners for saddle point problems have been proposed, such as the block diagonal preconditoner [23]

$$P_{\mathrm{BDP}} := \begin{bmatrix} P_A & 0 \\ 0 & P_S \end{bmatrix}, \tag{1.2}$$

where P_A and P_S are the respective SPD matrices that approximate A and the Schur complement $S = C + B^T A^{-1} B$; the accelerated Hermitian and skew-Hermitian splitting (AHSS) [3,5] preconditioner

$$P_{\mathrm{AHSS}} := \frac{1}{\alpha+\beta} \begin{bmatrix} \alpha I + A & \\ & \beta I + C \end{bmatrix} \begin{bmatrix} \alpha I & B \\ -B^T & \beta I \end{bmatrix}; \tag{1.3}$$

and the triangular inexact constraint preconditioner [17]:

$$P_{\mathrm{TICP}} := \begin{bmatrix} P_A & \\ & -P_S \end{bmatrix} \begin{bmatrix} I & P_A^{-1} B \\ 0 & I \end{bmatrix}. \tag{1.4}$$

When $C = 0$, the splitting preconditioner (SP) is defined as [20]

$$P_{\mathrm{SP}} := \begin{bmatrix} A + \alpha B B^T & 0 \\ -2B & \frac{1}{\alpha} I \end{bmatrix}. \tag{1.5}$$

Benzi *et al.* [14] have surveyed numerical methods and preconditioners for saddle point problems.

By transforming the coefficient matrix in the linear system (1.1), in Section 2 we construct a lopsided alternating direction iteration (LADI) method that only requires a solver for two linear sub-systems of equations with SPD coefficient matrices, and obtain a new block diagonal preconditioner induced by the LADI method. The convergence of the new iteration method is analysed in Section 3, and some spectral properties of the preconditioned matrix are discussed in Section 4. Numerical experiments in Section 5 show the effectiveness of GMRES with the new preconditioner, and our concluding remarks are in Section 6.

2. The LADI Method

By transforming for the coefficient matrix, we now obtain a lopsided ADI (LADI) method for the generalised saddle point problem (1.1), together with a new block diagonal preconditioner induced by the LADI method.