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A Posteriori Error Estimates of Semidiscrete Mixed Finite Element Methods for Parabolic Optimal Control Problems

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Abstract. *A posteriori* error estimates of semidiscrete mixed finite element methods for quadratic optimal control problems involving linear parabolic equations are developed. The state and co-state are discretised by Raviart-Thomas mixed finite element spaces of order k, and the control is approximated by piecewise polynomials of order k ($k \geq 0$). We derive our *a posteriori* error estimates for the state and the control approximations via a mixed elliptic reconstruction method. These estimates seem to be unavailable elsewhere in the literature, although they represent an important step towards developing reliable adaptive mixed finite element approximation schemes for the control problem.

AMS subject classifications: 49J20, 65N30

Key words: A posteriori error estimates, optimal control problems, parabolic equations, elliptic reconstruction, semidiscrete mixed finite element methods.

1. Introduction

Optimal control problems (OCP) involving partial differential equations (PDE) arise in various fields such as fluid dynamics, environmental modelling and engineering. Efficient numerical methods are usually required to solve such OCP. Finite element approximations have proven suitable in engineering design work [4, 18–20, 23, 26, 35, 37], and very many authors have considered OCP governed by elliptic or parabolic state equations previously [1, 17, 22, 25, 27–30, 32, 36]. There has been a growing demand for reliable and efficient space-time algorithms to solve both linear and nonlinear time-dependent PDE numerically, and most of the algorithms are based on *a posteriori* error estimators in order to provide appropriate tools for adaptive mesh refinements. The theory for the *a posteriori* analysis of finite element methods for elliptic problems is well developed, but it is yet to be as complete for time-dependent linear and nonlinear problems. For parabolic problems, there are schemes dealing with space-time adaptivity [14, 15, 38, 39] or with time

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adaptivity alone [21], or with spatial adaptivity on keeping the temporal variable continuous [3,4]. For space-time adaptivity, typically the finite element discretization depends upon a space-time variational formulation and the error indicators include both space and time errors. Makridakis & Nochetto [33] introduced an elliptic reconstruction operator that plays a role in *a posteriori* estimates quite similar to the role played by an elliptic projection for recovering optimal *a priori* error estimates for parabolic problems [41]. This elliptic construction method was developed for a completely discrete scheme based on the backward Euler method [31], for maximum norm estimates [13], and for discontinuous Galerkin methods for parabolic problems [16].

In many control problems, the objective functional contains the gradient of the state variables. For example, in flow control problems the gradient representing the Darcy velocity is an important variable, and in temperature control problems large temperature gradients during cooling or heating are important as they may be quite destructive. The accuracy of the gradient is therefore important in the numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases, since both the scalar variable and its flux variable can be approximated to the same accuracy — e.g. see Ref. [6]. Indeed, when the objective functional contains the gradient of the state variable, mixed finite element methods can be used for the state equation such that both the scalar variable and its flux variable can be approximated to the same accuracy. Recently, we have worked on both *a priori* superconvergence and *a posteriori* error estimates in the application of mixed finite element methods to linear elliptic OCP [7,8,10].

In this article, we develop *a posteriori* error estimates of a semidiscrete mixed finite element approximation for parabolic OCP. Combining the elliptic construction idea of Ref. [33] with the parabolic OCP, we define a corresponding mixed elliptic construction for the state and co-state variables, and then use this mixed elliptic construction to derive *a posteriori* error estimates for both the state and the control approximation. The OCP of interest are of the form

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\|y-y_d\|^2+\|u\|^2)\,dt\right\}, \tag{1.1}$$

$$y_t(x,t)+\operatorname{div}\boldsymbol{p}(x,t)=f(x,t)+u(x,t), \qquad x\in\Omega,\ t\in J, \tag{1.2}$$

$$\boldsymbol{p}(x,t)=-A(x)\nabla y(x,t), \qquad x\in\Omega,\ t\in J, \tag{1.3}$$

$$y(x,t)=0, \qquad x\in\partial\Omega,\ t\in J, \tag{1.4}$$

$$y(x,0)=y_0(x), \qquad x\in\Omega, \tag{1.5}$$

where the bounded open set $\Omega\subset R^2$ is a convex polygon with boundary $\partial\Omega$ and $J=[0,T]$.

Let K be a closed convex set in control space $U=L^2(0,T;L^2(\Omega))$, $f, y_d\in L^2(0,T;L^2(\Omega))$, $\boldsymbol{p}_d\in(L^2(0,T;L^2(\Omega)))^2$ and $y_0\in H_0^1(\Omega)$. We assume the coefficient matrix

$$A(x)=\big(a_{ij}(x)\big)_{2\times 2}\in W^{1,\infty}(\bar\Omega;R^{2\times 2})$$

is symmetric 2×2, that there are constants $c_1,c_2>0$ for any vector $X\in R^2$ such that