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Efficient Numerical Solution of the Multi-Term Time Fractional Diffusion-Wave Equation

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Abstract. Some efficient numerical schemes are proposed to solve one-dimensional and two-dimensional multi-term time fractional diffusion-wave equation, by combining the compact difference approach for the spatial discretisation and an L1 approximation for the multi-term time Caputo fractional derivatives. The unconditional stability and global convergence of these schemes are proved rigorously, and several applications testify to their efficiency and confirm the orders of convergence.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Multi-term time fractional diffusion-wave equation, compact difference scheme, discrete energy method, convergence.

1. Introduction

Recently, fractional differential equations have been invoked in various applications. Unlike classical differential equations of integer order, where the derivatives depend only on the local behaviour of the function, fractional differential equations accumulate all of the information on the function in a weighted form. This is the so-called memory effect in physics, chemistry and other research areas — e.g. see Refs. [1–3] and references therein. In particular, the time fractional diffusion-wave equation models a wide range of important physical phenomena, including *inter alia* the propagation of mechanical waves in viscoelastic media [4], a non-Markovian diffusion process with memory [5], and charge transport in amorphous semiconductors [6].

Since analytical solutions are rare and to date restricted to simpler fractional partial differential equations, there has been increasing interest in the development of effective and easy to use numerical schemes. Yuste & Murillo [7,8] constructed difference schemes using an L1 discretisation formula for the fractional diffusion equation and an L2 discretisation formula for fractional diffusion-wave equations, respectively. The stability analysis of their

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schemes was carried out via the von Neumann method. Langlands & Henry [9] considered an implicit numerical scheme for a fractional diffusion equation, using the backward Euler approximation to discretise the first order time derivative and an L1 scheme to approximate the fractional order time derivative. Chen *et al.* [10] constructed a difference scheme based on the Grünwald-Letnikov formula. They also provided both an explicit and an implicit scheme for the two-dimensional anomalous sub-diffusion equation, using relationships between the fractional Grünwald-Letnikov and Riemann-Liouville definitions [11]. The corresponding theoretical analysis for stability and convergence was undertaken using the Fourier method, and a highly accurate algorithm was constructed exploiting Richardson extrapolation. Sun & Wu [12] derived two fully discrete difference schemes for the fractional diffusion-wave and sub-diffusion equations, and proved that the schemes are uniquely solvable, unconditionally stable, and respectively $\mathscr{O}(\tau^{3-\alpha}+h^2)$ and $\mathscr{O}(\tau^{2-\alpha}+h^2)$ convergent in the maximum norm. Recently, Zhang *et al.* [13] constructed a compact alternating direction implicit (ADI) scheme to solve two-dimensional time fractional diffusion-wave equations.

There has also been some previous work on the numerical solution of problems with multiple fractional derivatives. Diethelm & Luchko [14] gave an algorithm for solving the multi-term linear fractional differential equations based on Ref. [15], but their method may require a large amount of computational effort to calculate the associated weights. Edwards *et al.* [16] solved linear multi-term fractional differential equations through a reduction to a system of ordinary and fractional differential equations. Based on the analogue equation concept, Katsikadelis [17] presented a numerical method to solve linear multi-term fractional differential equations.

A key issue in solving fractional-order diffusion models numerically is the design of efficient algorithms for the space and time discretisation. The complexity of fractional differential equations is because the fractional derivatives are nonlocal and characterised by historic dependence and universal mutuality. Thus all previous solutions must be saved to compute the solution at the current time level, which makes the storage expensive. Due to their high spatial accuracy, compact difference methods need few grid points to produce accurate solutions. However, there appear to be very few previous studies on efficient numerical methods for problems involving multi-term fractional derivatives.

This article provides some numerical schemes to solve the one-dimensional and two-dimensional multi-term fractional differential equations of the general form (cf. [17–19])

$$P({}^{C}\mathscr{D}_t)u(\mathbf{X},t)=\kappa\Delta u(\mathbf{X},t)+f(\mathbf{X},t)\,,\quad \mathbf{X}\in\Omega\,,\ 0<t\leqslant T\,, \tag{1.1}$$

where κ is a positive diffusion constant. The multi-term fractional operator $P({}^{C}\mathscr{D}_t)$ is defined by

$$P({}^{C}\mathscr{D}_t)v(\mathbf{X},t)=\left({}_{0}^{C}\mathscr{D}_t^{\alpha}+\sum_{i=1}^{s}a_i\,{}_{0}^{C}\mathscr{D}_t^{\alpha_i}\right)v(\mathbf{X},t)\,,$$

where $1<\alpha_s<\cdots<\alpha_1<\alpha<2$, $a_i>0$, $i=1,2,\cdots,s$, and

$${}_{0}^{C}\mathscr{D}_t^{\alpha}v(t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{v''(s)}{(t-s)^{\alpha-1}}ds$$