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Two-Grid Finite Element Methods for the Steady Navier-Stokes/Darcy Model

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Abstract. Two-grid finite element methods for the steady Navier-Stokes/Darcy model are considered. Stability and optimal error estimates in the H^1-norm for velocity and piezometric approximations and the L^2-norm for pressure are established under mesh sizes satisfying $h = H^2$. A modified decoupled and linearised two-grid algorithm is developed, together with some associated optimal error estimates. Our method and results extend and improve an earlier investigation, and some numerical computations illustrate the efficiency and effectiveness of the new algorithm.

AMS subject classifications: 65N15, 65N30, 76D07

Key words: Navier-Stokes equations, Darcy's law, multimodeling problems, two-grid method.

1. Introduction

A two-grid method is an efficient numerical scheme for partial differential equation (PDE) based on two spaces with different meshes, first introduced by Xu [26, 27] for both linear and nonlinear elliptic PDE. Two-grid schemes have since been studied by many researchers. For example, Dawson *et al.* [6, 7] studied nonlinear parabolic equations using both finite element and finite difference methods. For the Navier-Stokes equations, we refer to Refs. [14, 17–21] and references therein. Recently, we have studied the stability and convergence of a two-grid finite volume method for nonlinear parabolic problems in semi-discrete and fully discrete formulations — cf. Refs. [28, 29, 31], respectively.

In recent years, the coupling of incompressible fluid flow with porous media flows has also been researched extensively [8–10, 15, 22, 30]. The fluid flow and the porous media flow are respectively modelled by the Navier-Stokes equations and Darcy's law, with the interface coupled via certain conditions. Several numerical schemes have been proposed for this model [5, 11], but any implementation for the coupled nonlinear discrete problem is

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usually complicated and difficult. In order to solve the Navier-Stokes/Darcy model model efficiently, Cai *et al.* [4] developed a decoupled linearised method, and established the following convergence result for velocity and pressure:

$$\begin{aligned}&\|\mathbf{u}_f-\mathbf{u}_{fh}^d\|_{(H^1(\Omega_f))^d}+\|p_f-p_{fh}^d\|_{L^2(\Omega_f)}\\ \leq& C(h+H^{3/2})\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big)\,. \end{aligned} \tag{1.1}$$

Here and below, C (with or without a subscript) denotes a positive constant, independent of the two mesh sizes h and H where $h \ll H$. The estimates for $\mathbf{u}_f$ and p_f are not optimal, as suggested by numerical experiments [4], and the estimate (1.1) might be improved to $O(H^2)$. This motivated us to propose our modified decoupled two-grid finite element numerical scheme, and derive the optimal estimates of $O(h)$ for both $\mathbf{u}_f$ and p_f with $H = \sqrt{h}$.

The classical two-grid technique we adopt to treat the steady Navier-Stokes/Darcy problem involves: (I) a coupled and nonlinear problem on a coarse grid with mesh size H; and (II) a coupled and linear problem on fine mesh with mesh size $h = H^2$, using Newton iteration to linearise the nonlinear term. Consequently, in lieu of (1.1) we have

$$\begin{aligned}&\|\mathbf{u}_f-\mathbf{u}_{fh}^c\|_{(H^1(\Omega_f))^d}+\|\phi-\phi_h^c\|_{H^1(\Omega_p)}+\|p_f-p_{fh}^c\|_{L^2(\Omega_f)}\\ \leq& C(h+H^{2})\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big)\,. \end{aligned} \tag{1.2}$$

Secondly, we develop a modified decoupled and linearised two-grid algorithm. Our numerical scheme is thus: (I) a coupled Navier-Stokes/Darcy model to be solved on the coarse mesh; (II) a Darcy problem with the coarse grid approximation $(\mathbf{u}_{fH}, p_H, \psi_H)$ to the interface coupling conditions on the fine mesh; and (III) a linearised Navier-Stokes problem treated with the numerical solution of the Darcy problem on the fine mesh and the interface coupling conditions. Consequently, we then obtain

$$\begin{aligned}&\|\mathbf{u}_f-\mathbf{u}_{fh}^{md}\|_{(H^1(\Omega_f))^d}+\|p_f-p_{fh}^{md}\|_{L^2(\Omega_f)}\\ \leq& C(h+H^{2})\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big)\,. \end{aligned} \tag{1.3}$$

Comparing (1.1) with (1.2) and (1.3), in using the two mesh sizes satisfying $h = H^2$ we evidently have improved the error estimates in Ref. [4] for both $\mathbf{u}_f$ and p_f.

The rest of this article is organised as follows. The coupled steady Navier-Stokes/Darcy model and associated properties are described in Section 2. In Section 3, the coupled two-grid finite element method is developed and its convergence is established. In Section 4, the decoupled linearised two-grid scheme in Ref. [27] is reviewed and our modified decoupled and linearised algorithm to improve the computational efficiency is proposed. Some enhancements of our decoupled two-grid method are discussed in Section 5, and numerical results that verify the performance of our developed numerical schemes are presented in Section 6. Brief concluding remarks are made in Section 7.