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A New GSOR Method for Generalised Saddle Point Problems

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Abstract. A novel generalised successive overrelaxation (GSOR) method for solving generalised saddle point problems is proposed, based on splitting the coefficient matrix. The proposed method is shown to converge under suitable restrictions on the iteration parameters, and we present some illustrative numerical results.

AMS subject classifications: 65F10, 65F50

Key words: Generalised saddle point problem, generalised successive overrelaxation method, splitting, convergence analysis.

1. Introduction

The generalised saddle point problem considered here has the form

$$\begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $C \in R^{n\times n}$ is a symmetric positive semidefinite matrix, $B \in R^{m\times n}$ is a matrix of full column rank where $m \geq n$ and the superscript T denotes its transpose, and $b \in R^m$ and $q \in R^n$ are given vectors. Linear systems of the form (1.1) arise in a variety of scientific and engineering applications, including mixed or hybrid finite element approximations of second-order elliptic problems [5,42], computational fluid dynamics [19,21,22,25], least squares problems [2], inversion of geophysical data [32], stationary semiconductor device [45,47], elasticity problems and Stokes equations [5].

In recent years, many iterative methods have been introduced to solve the problem (1.1), including Uzawa-type schemes [14, 20, 23, 25, 33, 34, 51, 53], iterative projection methods [3], block and approximate Schur complement preconditioners [17,19,22,35,40, 42,43], iterative null space methods [1,26,48], splitting methods [4,7,9–13,18,29,30,36, 38,39,41,46,50], indefinite preconditioning [31,37], and preconditioning methods based

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on approximate factorisation of the coefficient matrix [6,8,28,44]. A classical approach to solve (1.1) is the successive overrelaxation (SOR) iteration method [49], which can involve relatively low computation per iterative step. However, the SOR method requires a good or optimal iteration parameter to achieve comparable rates of convergence. To address this, Bai *et al.* [15] proposed a generalised SOR method, where another parameter is introduced for solving the problem (1.1) when $C = 0$. Bai & Wang [16] then developed parameterised inexact Uzawa methods to solve large sparse generalised saddle point problems via the generalised SOR method, where another symmetric positive definite matrix is introduced. Based on the SOR-like methods, Feng & Shao [24] proposed a generalised SOR-like method by introducing uncertain parameters, and Guo *et al.* [27] considered a new splitting of the coefficient matrix in a modified SOR-like method for solving the system (1.1) with $C = 0$. Zhang & Lu [52] established a generalised symmetric SOR method based on the well-known symmetric SOR iteration method for the saddle point problem.

However, all of the SOR-like methods mentioned above need to solve a linear algebraic system each step, which is difficult and time-consuming. In this article, we propose a new generalised successive overrelaxation method for the generalised saddle point problem (1.1) based on splitting the matrix A. Thus instead of solving the linear system with the large coefficient matrix A, we only need to solve a system with a triangular matrix in each step. Moreover, we shall show the convergence of our method under suitable restrictions on the iteration parameters.

Throughout , we use the following notation: $R^{m\times n}$ and $C^{m\times n}$ are the set of $m \times n$ real and complex matrices; $R^m = R^{m\times 1}$ and $C^m = C^{m\times 1}$. $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; and for $H \in R^{n\times n}$, we write H^{-1}, rank(H), $\mathscr{N}\{H\}$, $\mathscr{R}\{H\}$, $\Lambda(H)$ and $\rho(H)$ to denote the inverse, the rank, the null space, the image space, the spectrum and the spectral radius of the matrix H, respectively. For $x \in C^n$, x^* and $\|x\|$ respectively denote the conjugate transpose and the norm of the vector x, and I_l denotes the identity matrix of order l.

The organisation of this paper is as follows. In Section 2, we present our new generalised successive overrelaxation method for solving generalised saddle point problems. We discuss its convergence in Section 3, present the results of numerical experiments in Section 4 to show the effectiveness of our method, and then briefly summarise our conclusions in Section 5.

2. The Generalised SOR Method

In this section, we consider a new generalised successive overrelaxation method for solving the generalised saddle point problems (1.1). Firstly, we split the matrix A into the following form:

$$A = D - L - L^T, \tag{2.1}$$

where $D = \mathrm{diag}\{a_{11}, a_{22}, \cdots, a_{mm}\}$ is the diagonal matrix incorporating the diagonal entries of A, and $-L$ is the strictly lower triangular part of the matrix A. Thus the coefficient matrix