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Admissible Regions for Higher-Order Finite Volume Method Grids

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Abstract. Admissible regions for higher-order finite volume method (FVM) grids are considered. A new Hermite quintic FVM and a new hybrid quintic FVM are constructed to solve elliptic boundary value problems, and the corresponding admissible regions are investigated. A sufficient condition for the uniform local-ellipticity of the new hybrid quintic FVM is obtained when its admissible region is known. In addition, the admissible regions for a large number of higher-order FVMs are provided. For the same class of FVM (Lagrange, Hermite or hybrid), the higher order FVM has a smaller admissible region such that stronger geometric restrictions are required to guarantee its uniform local-ellipticity.

AMS subject classifications: 65N30, 65N12

Key words: Finite volume method, admissible region, uniform local-ellipticity.

1. Introduction

Finite volume methods (FVM) are often invoked to solve partial differential equations numerically [2, 3, 17–19, 23]. The preservation of certain local conservation laws [14, 22] and flexible algorithm constructions [5, 7, 8, 12, 15, 16]) are attractive advantages, and the numerical solution of the linear system that arise have been addressed [10, 20]. Moreover, a high order FVM scheme [5–8, 19, 21, 23] is helpful in developing the *hp*-version of a FVM. High order FVM on rectangular meshes have previously been considered [4, 24–26], and here we focus on high order FVM on triangular meshes.

A FVM can be viewed as a special type of Petrov-Galerkin method where the chosen trial space is a standard finite element space. The test space is not the same as the trial space for the FVM, as the uniform local-ellipticity of the family of discrete bilinear forms may

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be destroyed to some degree. For a linear FVM, the uniform local-ellipticity of the family of the discrete bilinear forms does hold without additional requirements on the primary triangulation [19]. However, for higher-order FVM certain geometric requirements on the shapes of the triangles in the primary triangular mesh are needed to establish the uniform local-ellipticity [5–8, 19, 21, 23]. For example, in Ref. [7] it is shown that uniform local-ellipticity holds if $\theta_{\min} \geq 9.98°$ for the quadratic FVM scheme proposed in Ref. [21], where $\theta_{\min}$ denotes the minimal angle of the triangles in the triangulation.

The key point for the investigation of uniform local-ellipticity lies in establishing the admissible region, which we now illustrate. Letting Ω denote a polygonal domain in $\mathbb{R}^2$ and assuming $f \in L^2(\Omega)$, we consider the FVM on a triangular mesh to solve the Poisson equation

$$-\Delta u = f \tag{1.1}$$

in Ω for the unknown function u subject to the Dirichlet boundary condition, where Δ is the Laplacian operator. Let $\mathbb{N}_m := \{1, 2, \cdots, m\}$ for a positive integer m, and denote the three edges of a triangle K by ℓ_i, $i \in \mathbb{N}_3$, where $|\ell_i|$ is the length of the edge ℓ_i and without loss of generality we have $|\ell_1| \geq |\ell_2| \geq |\ell_3|$. Letting $T := \{K\}$ denote a triangulation of Ω, for each triangle $K \in T$ we introduce the two geometric parameters

$$r_{1,K} := |\ell_2|^2/|\ell_1|^2, \qquad r_{2,K} := |\ell_3|^2/|\ell_1|^2. \tag{1.2}$$

A family $\mathscr{T} := \{T\}$ of triangulations of Ω is said to be *regular* if there exists a positive constant $\theta_{\inf}$ such that

$$\theta_{\min,K} \geq \theta_{\inf} \ \ \forall\, K \in \bigcup_{T\in\mathscr{T}} T, \tag{1.3}$$

where $\theta_{\min,K}$ denotes the minimum angle of the triangle K.

When possibly weaker requirements for boundary triangles intersecting the boundary of Ω are ignored, a necessary and sufficient condition on uniform local-ellipticity obtained as a corollary in Ref. [8] is as follows.

Proposition 1.1. *If a family $\mathscr{T}$ of triangulations of Ω is regular, then the corresponding family of discrete bilinear forms is uniformly local-elliptic if and only if there exists a compact subset $\mathbb{G}_0$ of the admissible region such that for all $T \in \mathscr{T}$ and all $K \in T$, $(r_{1,K}, r_{2,K}) \in \mathbb{G}_0$.*

Thus the uniform local-ellipticity corresponds to geometric requirements on the triangle shapes in the primary triangular mesh, where a larger admissible region implies weaker geometric requirements (and conversely). Moreover, when the admissible region is known, convenient sufficient conditions can be derived for the uniform local-ellipticity by choosing a proper $\mathbb{G}_0$. In brief, the admissible region is vital for the establishment of the uniform local-ellipticity, which leads to the optimal error estimate for the FVM.

In this article, we construct a new Hermite quintic FVM and a new hybrid quintic FVM. For the new Hermite quintic FVM grid, the admissible region is shown to be empty so that its uniform local-ellipticity does not hold no matter how regular the triangulation may be. The admissible region for the new hybrid quintic FVM grid is determined, and a convenient sufficient condition for its uniform local-ellipticity is derived.