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A Finite Difference Method for Boundary Value Problems of a Caputo Fractional Differential Equation

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Abstract. In this paper, we consider a two-point boundary value problem with Caputo fractional derivative, where the second order derivative of the exact solution is unbounded. Based on the equivalent form of the main equation, a finite difference scheme is derived. The L_∞ convergence of the difference system is discussed rigorously. The convergence rate in general improves previous results. Numerical examples are provided to demonstrate the theoretical results.

AMS subject classifications: 65L10, 65L12, 65L20

Key words: Boundary value problem, Caputo fractional derivative, Finite difference method, Convergence, Derivative bound.

1. Introduction

Study on fractional derivatives has attracted lots of attentions in recent years due to its variety in applications. Significant progress on numerical analysis of differential equations with fractional-order derivatives have been made by many researchers [1, 3–6, 9–18, 20, 21, 24–29]. Study of efficient numerical methods for time-fractional differential equations grows to be a popular research topic in the past decade, interested readers can refer to some typical articles [1,16,17,24,28] where equations with Caputo derivative are considered. For space-fractional differential equations, approaches such as the Grünwald-Letnikov formula, shifted Grünwald-Letnikov formula, weighted and shifted Grünwald-Letnikov formula and compact operator were successfully applied to obtain numerical schemes for solving the Riemann-Liouville fractional differential equations, see also [3, 10, 18, 21, 25, 29] for more details. However, many of these works were done under the special restriction that the

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exact solutions are smooth on the domain where they are posed and this assumption may not hold in some practical applications. Therefore singular behavior, which may be the true behavior [19], of solutions near boundaries deserves further investigation and this is the main concern of our current study.

In this paper, we consider the following two-point boundary value problem

$$-{}_0^C D^\gamma u(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

subject to

$$u(0) - \alpha_0 u'(0) = g_0, \quad u(1) + \alpha_1 u'(1) = g_1, \tag{1.2}$$

where $\gamma \in (1,2)$ and ${}_0^C D^\gamma u(x)$ is the Caputo fractional derivative of $u(x)$ defined as

$${}_0^C D^\gamma u(x) = \frac{1}{\Gamma(2-\gamma)} \int_0^x u''(s)(x-s)^{1-\gamma} ds,$$

with $\Gamma(\cdot)$ being the gamma function. The constants $\alpha_0, \alpha_1, g_0, g_1$ and the functions b, c, f are given, and α_0, α_1 satisfy

$$\alpha_0 \geq \frac{1}{\gamma - 1}, \quad \alpha_1 \geq 0. \tag{1.3}$$

We note that the condition was proposed in [2]. Very recently, finite difference methods for (1.1)–(1.3) were studied in [7, 23], where rigorous analysis under reasonably general and realistic hypotheses on the behavior of the solution near the boundaries were given. It was showed that proposed finite difference schemes converge with order $h^{\gamma-1}$, where h is the grid size. The Caputo fractional derivative in these two references is approximated by the widely-used first order discretisation formula introduced in [22]. In [23], the convection term is approximated by upwind scheme, while the standard central difference is discussed in [7]. Numerical results in [7] showed that the central approximation is a little bit more accurate than the upwind approximation, but it will be restrictive when γ is close to 1. Further discussions on the numerical behavior of several finite difference schemes for some Riemann-Liouville and Caputo boundary value problems were given in [8]. In this paper, we focus on the Caputo fractional boundary value problem. One may note that the numerical experiments in [7, 23] showed that the schemes preform much better in practise and there may be rooms for improvement of the theoretical analysis. Inspired by these results, this paper continues the study in this direction. We propose to solve an equivalent form of (1.1)-(1.3). Following the delicate estimates established in [7, 23], we obtain improved convergence under an additional condition. We remark that the central and upwind difference approximations of the convection term make no essential difference to our proof. However, more routine arguments will be needed for the discussion of the central difference approximation. Therefore, for simplicity of presentation, we only discuss the upwind scheme in this paper.

Based on an equivalent form of the considered model, a finite difference scheme will be proposed in the next section. Error analysis of the proposed scheme will be studied in Section 3. In Section 4, numerical experiments are carried out to support the theoretical results, the paper ends with a brief conclusion.