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Convergence Analysis for a Three-Level Finite Difference Scheme of a Second Order Nonlinear ODE Blow-Up Problem

Chien-Hong Cho* and Chun-Yi Liu

Department of Mathematics, National Chung Cheng University, Min-Hsiung, Chia-Yi 621, Taiwan.

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Abstract. We consider the second order nonlinear ordinary differential equation $u''(t) = u^{1+\alpha}$ $(\alpha > 0)$ with positive initial data $u(0) = a_0$, $u'(0) = a_1$, whose solution becomes unbounded in a finite time T. The finite time T is called the blow-up time. Since finite difference schemes with uniform meshes can not reproduce such a phenomenon well, adaptively-defined grids are applied. Convergence with mesh sizes of certain smallness has been considered before. However, more iterations are required to obtain an approximate blow-up time if smaller meshes are applied. As a consequence, we consider in this paper a finite difference scheme with a rather larger grid size and show the convergence of the numerical solution and the numerical blow-up time. Application to the nonlinear wave equation is also discussed.

AMS subject classifications: 65L12

Key words: Blow-up, numerical blow-up time, finite difference method, nonlinear ODE.

1. Introduction

In this paper, we consider a second order nonlinear ordinary differential equation

$$u''(t) = u^{1+\alpha}(t), \qquad u(0) = a_0 > 0, \qquad u'(0) = a_1 > 0, \tag{1.1}$$

and its finite difference analogue. Here, $\alpha > 0$ is a parameter and $'$ denotes the differentiation. It is easy to show that the solution of (1.1) blows up in finite time T. In fact, multiplying the first equation of (1.1) by $u'(t)$, we have

$$u'(t) = \left(\frac{2}{2+\alpha}u^{2+\alpha} + C\right)^{\frac{1}{2}}, \tag{1.2}$$

where $C = a_1^2 - 2a_0^{2+\alpha}/(2+\alpha)$. Therefore, for $a_0, a_1 > 0$, the solution becomes unbounded in finite time $T = \int_{a_0}^{\infty} ds/g(s)$, where

*Corresponding author. *Email address:* `chcho20@ccu.edu.tw` (C.-H. Cho)

$$g(s)=\left(\frac{2}{2+\alpha}s^{2+\alpha}+C\right)^{\frac{1}{2}}. \tag{1.3}$$

This phenomenon is known as blow-up and the finite time T is called the blow-up time.

It is known that a scheme with uniform time meshes can not reproduce the finite-time blow-up phenomenon and thus adaptively-defined time meshes are considered to be necessary for such problems. See for instance [1, 4, 8, 11, 18]. Consequently, we consider the following finite difference analogue of (1.1)

$$\frac{1}{\tau_n}\left(V^{n+1}-V^n\right)=(U^n)^{1+\alpha} \quad \text{and} \quad V^n=\frac{U^n-U^{n-1}}{\Delta t_{n-1}}, \tag{1.4}$$

where the grid size Δt_n is given adaptively by

$$\Delta t_n=\tau\cdot\min\left\{1,\frac{1}{(U^n)^\gamma}\right\} \quad \left(0<\gamma<\frac{\alpha}{2}\right). \tag{1.5}$$

Here, τ is a prescribed constant and $\tau_n=(\Delta t_{n-1}+\Delta t_n)/2$. $t_0=0$ and $t_n=t_{n-1}+\Delta t_{n-1}$ ($\forall n\geq 1$) denote the grid points. The discrete initial data is given by

$$U^0=a_0>0 \quad \text{and} \quad \frac{U^1-U^0}{\Delta t_0}=a_1>0. \tag{1.6}$$

We now set the numerical blow-up time $T(\tau)$ by

$$T(\tau)=\lim_{n\to\infty}t_n=\sum_{n=0}^{\infty}\Delta t_n.$$

Then we are going to prove in this paper

Theorem 1.1. *Let $\{U^n\}$ be the solution of* (1.4)(1.6). *Let T denote the blow-up time of the solution of* (1.1) *and let T_0 be an arbitrary number such that $0<T_0<T$. Then there exist positive constant C and τ_0, depending only on T_0 and the initial data, such that*

$$\left|U^n-u(t_n)\right|\leq C\tau$$

holds so far as $t_n\leq T_0$ and $0<\tau\leq\tau_0$.

Theorem 1.2. *Let $\{U^n\}$ be the solution of* (1.4)(1.6). *Let T denote the blow-up time of* (1.1). *Then we have*

$$T(\tau)\to T \quad \text{as} \quad \tau\to 0.$$

The conclusions of Theorem 1.1 and 1.2 themselves are not important. As a matter of fact, we may reduce the second order nonlinear ODE (1.1) to the first order ODE (1.2) and then apply the method given in [8] for the computation of the numerical solution and the