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Three-Layer Non-hydrostatic Staggered Scheme for Free Surface Flow

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Abstract. In this paper, a finite difference algorithm using a three-layer approximation for the vertical flow region to solve the 2D Euler equations is considered. In this algorithm, the pressure is split into hydrostatic and hydrodynamic parts, and the predictor-corrector procedure is applied. In the predictor step, the momentum hydrostatic model is formulated. In the corrector step, the hydrodynamic pressure is accommodated after solving the Laplace equation using the Successive Over Relaxation (SOR) iteration method. The resulting algorithm is first tested to simulate a standing wave over an intermediate constant depth. Dispersion relation of the scheme is derived, and it is shown to agree with the analytical dispersion relation for $kd < \pi$ with 94% accuracy. The second test case is a solitary wave simulation. Our computed solitary wave propagates with constant velocity, undisturbed in shape, and confirm the analytical solitary wave. Finally, the scheme is tested to simulate the appearance of the undular bore. The result shows a good agreement with the result from the finite volume scheme for the Boussinesq-type model by Soares-Frazão and Guinot (2008).

AMS subject classifications: 65M10, 78A48

Key words: The 2D Euler equations, non-hydrostatic scheme, solitary wave, undular bore.

1. Introduction

Free surface flows are mostly described using the shallow water equations (SWE) with hydrostatic pressure distribution and depth-averaged velocity assumptions. With that assumptions, SWE cannot be used to describe the secondary wave phenomena, such as an undular bore. Undular bore is a dispersive wave phenomenon observed in nature. Kim and Lynett [2] examine undular bores and shocks generated by dam-break flows or tsunamis using the 2D Euler equations that account for the hydrodynamic pressure term.

This paper aims to develop a numerical model to simulate water wave dynamics that involve dispersion effects. Here, we consider the 2D Euler equations applicable to a column

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of water, assumed as an inviscid ideal fluid of constant temperature. Since we are dealing with free surface flows, an extra equation for the free surface condition is needed, and that comes from a depth integrated continuity equation after incorporating the kinematic boundary equation. For the sake of numerical computation efficiency, the vertical axis on the Euler equations is resolved only by a few cells. In this paper, we focus on the three layer approximation of the 2D Euler model.

The organization of this paper is as follows. The first two sections discuss the governing equations and the formulation of the three-layer staggered grid scheme. The discussion is followed by the numerical dispersion relation and its comparison with the analytical dispersion relation. In Section 4 the scheme is validated with three benchmark tests: standing wave, solitary wave, and tidal bore with undulations. Those are test cases for the non-hydrostatic numerical schemes. Conclusions are outlined in the last section.

2. The Governing Equations

We consider a physical domain that is bounded on top by the free surface $z = \eta(x,t)$ and the bottom $z = -d(x)$. The 2D Euler equations for free surface flows in $x-$ and $z-$direction with velocity components u and w as the dependent variables are given by the momentum and continuity equations as noted in [1], i.e.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.3}$$

where $p(z,x,t)$ is the pressure, ρ the fluid density, and g the acceleration due to gravity. For horizontally dominant flow, in which the vertical velocity is high order, then (2.1), (2.2), and (2.3) can be simplified to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2.4}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{2.5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2.6}$$

Since we are examining the free surface flows, we need an additional equation for determining water level $\eta(x,t)$. The equation is obtained by integrating the continuity equation (2.6) over the water depth $h = \eta + d$, i.e.

$$\int_{-d}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz = 0. \tag{2.7}$$