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Approximation by Nörlund Means of Hexagonal Fourier Series

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Abstract. Let f be an $H-$periodic Hölder continuous function of two real variables. The error $\|f-N_n(p;f)\|$ is estimated in the uniform norm and in the Hölder norm, where $p=(p_k)_{k=0}^{\infty}$ is a nonincreasing sequence of positive numbers and $N_n(p;f)$ is the nth Nörlund mean of hexagonal Fourier series of f with respect to $p=(p_k)_{k=0}^{\infty}$.

Key Words: Hexagonal Fourier series, Hölder class, Nörlund mean.
AMS Subject Classifications: 41A25, 41A63, 42B08

1 Introduction

In general, approximation problems of functions of several real variables defined on cubes of the Euclidean space are studied by assuming that the functions are periodic in each of their variables (see, for example [10, Sections 5.3 and 6.3] and [12, Vol II, Chapter XVII]). But in the case of non tensor-product domains, for example in hexagonal domains of $\mathbb{R}^2$, another definition of periodicity is needed. For such domains most useful periodicity is the periodicity defined by lattices. We refer to [5] for general information about lattices.

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2},\frac{1}{2}\right)^2$, the simplest lattice is the hexagonal lattice and the simplest spectral set is the regular hexagon.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H=\begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

and

$$\Omega_H=\left\{(x_1,x_2)\in\mathbb{R}^2:\ -1\le x_2,\ \frac{\sqrt{3}}{2}x_1\pm\frac{1}{2}x_2<1\right\}.$$

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It is more convenient to use the homogeneous coordinates (t_1,t_2,t_3) that satisfy $t_1+t_2+t_3=0$. If we define

$$t_1:=-\frac{x_2}{2}+\frac{\sqrt{3}x_1}{2}, \quad t_2:=x_2, \quad t_3:=-\frac{x_2}{2}-\frac{\sqrt{3}x_1}{2}, \tag{1.1}$$

the hexagon Ω_H becomes

$$\Omega=\left\{(t_1,t_2,t_3)\in\mathbb{R}^3:\ -1\le t_1,t_2,-t_3<1,\ t_1+t_2+t_3=0\right\},$$

which is the intersection of the plane $t_1+t_2+t_3=0$ with the cube $[-1,1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1+t_2+t_3=0$, that is

$$\mathbb{R}^3_H=\left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3: t_1+t_2+t_3=0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H=\mathbb{Z}^3\cap\mathbb{R}^3_H$.

A function $f:\mathbb{R}^2\to\mathbb{C}$ is called $H-$periodic if

$$f(x+Hk)=f(x)$$

for all $k\in\mathbb{Z}^2$ and $x\in\mathbb{R}^2$. If we define $\mathbf{t}\equiv\mathbf{s}\ (mod 3)$ as

$$t_1-s_1\equiv t_2-s_2\equiv t_3-s_3\ (mod 3)$$

for $\mathbf{t}=(t_1,t_2,t_3)$, $\mathbf{s}=(s_1,s_2,s_3)\in\mathbb{R}^3_H$, it follows that the function f is $H-$periodic if and only if $f(\mathbf{t})=f(\mathbf{t}+\mathbf{s})$ whenever $\mathbf{s}\equiv\mathbf{0}\ (mod 3)$. It is clear that

$$\int_\Omega f(\mathbf{t}+\mathbf{s})\,d\mathbf{t}=\int_\Omega f(\mathbf{t})\,d\mathbf{t}, \quad \left(\mathbf{s}\in\mathbb{R}^3_H\right),$$

holds for $H-$periodic integrable function f (see [11]).

$L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f,g\rangle_H:=\frac{1}{|\Omega|}\int_\Omega f(\mathbf{t})\overline{g(\mathbf{t})}\,d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω. The functions

$$\phi_{\mathbf{j}}(\mathbf{t}):=e^{\frac{2\pi i}{3}\langle\mathbf{j},\mathbf{t}\rangle}, \quad \left(\mathbf{t}\in\mathbb{R}^3_H\right),$$

are $H-$periodic and by a theorem of B. Fuglede (see [2]) the set

$$\left\{\phi_{\mathbf{j}}(\mathbf{t}):\mathbf{j}\in\mathbb{Z}^3_H\right\}$$

becomes an orthonormal basis of $L^2(\Omega)$ (see also [5]).