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Some Sharpening and Generalizations of a Result of T. J. Rivlin

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Abstract. Let $p(z)=a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n$ be a polynomial of degree n. Rivlin [12] proved that if $p(z)\neq 0$ in the unit disk, then for $0<r\leq 1$,

$$\max_{|z|=r}|p(z)|\geq\left(\frac{r+1}{2}\right)^n\max_{|z|=1}|p(z)|.$$

In this paper, we prove a sharpening and generalization of this result and show by means of examples that for some polynomials our result can significantly improve the bound obtained by the Rivlin's Theorem.

Key Words: Inequalities, polynomials, zeros.

AMS Subject Classifications: 15A18, 30C10, 30C15, 30A10

1 Introduction

Let

$$p(z)=\sum_{j=0}^{n}a_jz^j$$

be a polynomial of degree n,

$$M(p,r):=\max_{|z|=r}|p(z)|,\quad r>0,\quad \|p\|:=\max_{|z|=1}|p(z)|,$$

and

$$D(0,K):=\{z:|z|<K\},\quad K>0.$$

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Then it is well known that

$$M(p',1) \le n\|p\|, \tag{1.1}$$

and

$$M(p,R) \le R^n \|p\|, \quad R \ge 1. \tag{1.2}$$

The above inequalities are known as Bernstein inequalities and have been the starting point of a considerable literature in approximation theory. Several papers and research monographs have been written on this subject (see, for example Govil and Mohapatra [3], Milovanović, Mitrinović and Rassias [6], Rahman [9], Nwaeze [7] and Rahman and Schmeisser [10, 11]).

For polynomials of degree n not vanishing in the interior of the unit circle, the above inequalities have been replaced by:

$$M(p',1) \le \frac{n}{2}\|p\|,$$

and

$$M(p,R) \le \Big(\frac{R^n+1}{2}\Big)\|p\|, \quad R \ge 1.$$

If one applies Inequality (1.2) to the polynomial $P(z) = z^n p(1/z)$ and use maximum modulus principle, one easily gets

Theorem 1.1. *Let*

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

be a polynomial of degree n. Then for $0 < r \le 1$,

$$M(p,r) \ge r^n \|p\|. \tag{1.3}$$

Equality holds for $p(z) = \alpha z^n$, α being a complex number.

The above result is due to Varga [13] who attributes it to E. H. Zarantonello.

It was shown by Govil, Qazi and Rahman [4] that the inequalities (1.1), (1.2) and (1.3) are all equivalent in the sense that any of these inequalities can be derived from the other.

The analogue of Inequality (1.3) for polynomials not vanishing in the interior of a unit circle was proved in 1960 by Rivlin [12], who in fact proved.

Theorem 1.2 (see Rivlin [12])**.** *Let*

$$p(z) = \sum_{j=0}^{n} a_j z^j \ne 0$$

in $D(0,1)$. Then for $0 < r \le 1$,

$$M(p,r) \ge \Big(\frac{r+1}{2}\Big)^n \|p\|.$$