Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 6, pp. 1420-1437

DOI: 10.4208/aamm.2016.m1420
December 2017

A New L^2 Projection Method for the Oseen Equations

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Received 21 January 2016; Accepted (in revised version) 9 January 2017

Abstract. In this paper, a new type of stabilized finite element method is discussed for Oseen equations based on the local L^2 projection stabilized technique for the velocity field. Velocity and pressure are approximated by two kinds of mixed finite element spaces, $P_l^2-P_1$, $(l=1,2)$. A main advantage of the proposed method lies in that, all the computations are performed at the same element level, without the need of nested meshes or the projection of the gradient of velocity onto a coarse level. Stability and convergence are proved for two kinds of stabilized schemes. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N12, 65N30

Key words: Oseen equations, L^2 projection method, pressure projection method.

1 Introduction

As a linearized model of the incompressible Navier-Stokes equations, the Oseen problem has attracted much research interest in the analysis of stabilized finite element methods. Mixed finite element methods for the Oseen equations must handle two numerical difficulties: compatibility of velocity and pressure spaces and advection dominated flows.

Stabilized finite element methods could conquer the lack of LBB stability. There are two approaches to design stabilized finite element methods. The first approach is based on the residual of the momentum equation, such as the multiscale enrichment method [2], the residual-free bubble method [18, 19], the least squares method [10, 11] and so on. Another approach is based on the projection stabilization, such as the pressure gradient projection (PGP) method (see [7, 8, 15]), the local pressure gradient stabilization (LPS) method [6] and the polynomial pressure projection stabilization (LPPS) method [9,16,29]. For PGP and LPS methods, the compressibility constraint is relaxed by subtracting the

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discontinuous pressure gradient from its projection onto a piecewise polynomial space. PGP method is based on the global L^2 projection, while LPS method is based on the local L^2 projection which can reduce the computations. LPS method adds terms of the form

$$\sum_{T\in\mathcal{T}_h}\psi_T h_T^2((I-\pi_{2h})\nabla p_l,(I-\pi_{2h})\nabla q_l)_T,$$

where p_l, q_l denote polynomials of degree less than l $(l\geq 1)$, $\psi_T>0$ the stabilized parameter, I the identity operator and π_{2h} the projection onto a coarse level. PGP and LPS methods are not easy to implement, since a special data structure of two-hierarchy mesh is required. As an alternative, LPPS method was introduced with the following term:

$$G_h(p_l,q_l)=\sum_{T\in\mathcal{T}_h}\theta_T((I-\pi_{l-1})p_l,(I-\pi_{l-1})q_l)_T, \tag{1.1}$$

where $\pi_{l-1}:L^2(\Omega)\rightarrow P_{l-1}^{dc}(\mathcal{T}_h)$ denotes the local L^2 projection, θ_T the stabilized parameter. In the method, all the computations are performed at the same element level, which simplify the computations. In particular, when pressure is approximated by piecewise linear polynomials, LPPS method's stabilization term has the following relationship:

$$(I-\pi_0)p_1|_T=(I-\pi_0)(\mathbf{x}\cdot\nabla p_1)|_T. \tag{1.2}$$

At present, the most popular approach to solve convection dominated cases is the variational multiscale (VMS) method (see [3,17,22–24,26–28,30,32] and so on), with the stabilized terms of the following form:

$$\sum_{T\in\mathcal{T}_h}\varpi_T((I-\mathcal{Q}_H)\nabla\mathbf{u}_h^l,(I-\mathcal{Q}_H)\nabla\mathbf{v}_h^l)_T$$

or

$$\sum_{T\in\mathcal{T}_h}\varpi_T(\nabla(I-\mathcal{Q}_H)\mathbf{u}_h^l,\nabla(I-\mathcal{Q}_H)\mathbf{v}_h^l)_T,$$

where ϖ_T is the stabilized parameter, $\mathbf{u}_h^l$, $\mathbf{v}_h^l$ denote polynomials of degree less than l $(l\geq 1)$, $\mathcal{Q}_H(H\geq h)$ is a projection onto a coarse level. Similar to LPS method, a special data structure of two-hierarchy mesh is required by VMS methods. Motivated by (1.2), the residual local projection (RELP) method [1,4] based on an enriching space strategy was proposed. Then, [5] used the additional terms of the RELP method and relaxed consistency to propose a local projection method which adds the following stabilized terms

$$\sum_{T\in\mathcal{T}_h}\frac{\delta_T}{\nu}(\chi_h(\mathbf{x}\cdot(\nabla\mathbf{u}_h^1)\beta),\chi_h(\mathbf{x}\cdot(\nabla\mathbf{v}_h^1)\beta))_T+\frac{\varrho_T}{\nu}(\chi_h(\beta\cdot\mathbf{x}\nabla\cdot\mathbf{u}_h^1),\chi_h(\beta\cdot\mathbf{x}\nabla\cdot\mathbf{v}_h^1))_T \tag{1.3}$$

to solve convection dominated, where δ_T and ϱ_T are the stabilized parameters, $\mathbf{u}_h^1$ is approximated by continuous piecewise linear polynomial, β is a advection field, $\chi_h:=$