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New Characterizations of Operator-Valued Bases on Hilbert Spaces

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Abstract. In this paper we study operator valued bases on Hilbert spaces and similar to Schauder bases theory we introduce characterizations of this generalized bases in Hilbert spaces. We redefine the dual basis associated with a generalized basis and prove that the operators of a dual g-basis are continuous. Finally we consider the stability of g-bases under small perturbations. We generalize two results of Krein-Milman-Rutman and Paley-Wiener [7] to the situation of g-basis.

Key Words: g-bases, dual g-bases, g-biorthogonal sequence.

AMS Subject Classifications: 41A58, 42C15

1 Introduction

The frame was first introduced by Duffin and Schaeffer [3] in the study of nonharmonic Fourier series in 1952. Frames are a generalization of the orthonormal bases in Hilbert spaces. Throughout this paper, $\mathcal{H}$, $\mathcal{K}$ are separable Hilbert spaces and I, J, J_i denote the countable (or finite) index sets and π_W denote the orthogonal projection of a closed subspace W of $\mathcal{H}$. We will always use $\mathcal{R}_T$ and $\mathcal{N}_T$ to denote range and the null spaces of an operator $T\in B(\mathcal{H},\mathcal{K})$ respectively. Recall that a family of vectors $\mathcal{F}=\{f_j\}_{j\in J}$ is called a frame for $\mathcal{H}$ if there exist constants $0<A\leq B<\infty$ such that

$$A\|f\|^2\leq\sum_{j\in J}|\langle f,f_j\rangle|^2\leq B\|f\|^2 \quad \text{for all } f\in\mathcal{H}. \tag{1.1}$$

The constants A and B are called frame bounds. If we only have the right-hand inequality of (1.1), $\mathcal{F}$ call a Bessel sequence. The representation space associated with a Bessel sequence $\mathcal{F}=\{f_j\}_{j\in J}$ is $\ell^2(J)$ and the synthesis operator of its is the bounded linear operator $T_{\mathcal{F}}:\ell^2(J)\to\mathcal{H}$ which defines by $T_{\mathcal{F}}(\{c_j\}_{j\in J})=\sum_{j\in J}c_jf_j$. The analysis operator for $\mathcal{F}$

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is $T_{\mathcal{F}}^*:\mathcal{H}\to\ell^2(J)$ which satisfies $T_{\mathcal{F}}^*f=\{\langle f,f_j\rangle\}_{j\in J}$. By composing $T_{\mathcal{F}}$ and $T_{\mathcal{F}}^*$ we obtain the frame operator

$$S_{\mathcal{F}}:\mathcal{H}\to\mathcal{H}\quad S_{\mathcal{F}}f=T_{\mathcal{F}}T_{\mathcal{F}}^*f=\sum_{j\in J}\langle f,f_j\rangle f_j,$$

which is a positive, self-adjoint and invertible operator. A Riesz basis for $\mathcal{H}$ is a family of the form $\{U(e_j)\}_{j\in J}$, where $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{H}$ and $U:\mathcal{H}\to\mathcal{H}$ is a bounded bijective operator. Nice properties of frames make them very useful in characterization of function spaces and other fields in sciences and engineering including coding theory, filter bank theory, signal and image processing and wireless communications. For more details about the theory and applications of frames and Riesz bases we refer the readers to [1,2,4].

Recently, W. Sun [6] introduced a g-frame and a Riesz g-basis for a Hilbert space and discussed some properties of them. In this paper we introduce new characterizations of g-basis and then we redefined the concepts of orthonormal g-basis and Riesz g-basis for a Hilbert space. We develop the basis theory to the situation of g-basis theory in Hilbert spaces.

The paper is organized as follows: Section 2, contains a new definition of g-basis for a Hilbert space. In this section similar to basis theory we first establishes a simple criterion for determining when a complete set of operators is a g-basis. Next we define the concepts of g-biorthogonal sequence, dual g-basis and obtain some characterizations of them. In Section 3, we redefine orthonormal g-basis and Riesz g-basis for a Hilbert space. We give some characterizations of orthonormal g-bases and Riesz g-bases. In Section 4, we study the stability of g-bases under small perturbations. We also generalize a result of Paley-Wiener [7] to the situation of g-basis.

2 Generalized Schauder bases

The purpose of this section is to investigate and develop the structure of bases for Hilbert spaces. We first define a g-basis for $\mathcal{H}$.

Definition 2.1. Let $\{W_j\}_{j\in J}$ be a sequence of closed subspaces of $\mathcal{K}$ and let $\Lambda_j\in B(\mathcal{H},W_j)$ be an onto operator for all $j\in J$. Then the family $\Lambda=\{\Lambda_j\}_{j\in J}$ is called a generalized Schauder basis or simply a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if for any $f\in\mathcal{H}$ there exists an unique sequence $\{g_j: g_j\in W_j\}_{j\in J}$ such that

$$f=\sum_{j\in J}\Lambda_j^*g_j,\tag{2.1}$$

with the convergence being in norm. If the series (2.1) converges unconditionally for each $f\in\mathcal{H}$, we say that Λ is an unconditional g-basis. Λ is called a g-basis for $\mathcal{H}$ with respect to $\mathcal{K}$ whenever $W_j=\mathcal{K}$ for all $j\in J$.