Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 2 (2017), pp. 149-156
DOI: 10.4208/ata.2017.v33.n2.5

The Oscillation Inequality of Harmonic Functions on Post Critically Finite Self-Similar Sets

Donglei Tang* and Rui Hu

Department of Applied Mathematics, Nanjing Audit University, Nanjing 211815, Jiangsu, China

Received 14 November 2016; Accepted (in revised version) 18 March 2017

Abstract. In this paper we establish the oscillation inequality of harmonic functions and Hölder estimate of the functions in the domain of the Laplacian on connected post critically finite (p.c.f.) self-similar sets.

Key Words: p.c.f. Self-similar sets, oscillation inequality, Hölder estimate, harmonic functions.

AMS Subject Classifications: 28A80

1 Introduction

Recently there are considerable interests in studying Hölder estimates of harmonic functions or functions in the domain of Dirichlet forms and the Laplacian on various fractal sets (see [1,2,4–10]). The results above can be used in the upper bound estimates of heat kernel, the transition density estimates, and spaces embedding.

In [6] Strichartz established the Hölder estimates of harmonic functions and the functions in the domain of the Laplacian on a class of Sierpinski gasket type sets with D_3 symmetry. In [9], the authors extended the results for harmonic functions on level n Sierpinski gaskets and n-gaskets.

Let F_i, $i=1,\cdots,N$ be contractive mappings, K is the unique nonempty compact set satisfying

$$K=\bigcup_{i=1}^{N} F_iK,$$

$V_0=\{p_1,\cdots,p_n\}$ is boundary of K with $n\le N$, and f is continuous on K. The initial energy of K can be defined as

$$\varepsilon_0(f,f)=\sum_{1\le i<j\le n}(f(p_i)-f(p_j))^2.$$

*Corresponding author. *Email addresses:* tdonglei@nau.edu.cn (D. L. Tang), hurui@nau.edu.cn (R. Hu)

In [4] Kigami showed that there existed the matrices A_j, $j=1,2,\cdots,N$ satisfying $h|_{F_iV_0}=A_jh|_{V_0}$ if h is a harmonic function on K.

For any stochastic matrix $M=(M_{i,j})$, Hajnal [3] defined

$$\lambda(M)=1-\min_{i_1,i_2}\sum_j \min\{M_{i_1,j},M_{i_2,j}\}\in[0,1].$$

Let $A_i=(a^i_{jk})_{n\times n}$, $i=1,2,\cdots,N$ denote the harmonic extension matrices obtained by using Kigami's Theory (see [4] for details). Let $\lambda_i\triangleq\lambda(A_i)$ for $i=1,2,\cdots,N$.

In the present paper we have obtained the main results as follows:

Theorem 1.1. *Let $Osc(f,E)$ denote the difference between the maximum and minimum values of f on a set E. If a continuous function h is harmonic on a p.c.f. self-similar set K, then*

$$Osc(h,F_iK)\le\lambda_i\cdot Osc(h,K)\quad \text{for}\ \ 1\le i\le N. \tag{1.1}$$

Moreover,

$$Osc(h,F_wK)\le\lambda_w\cdot Osc(h,K). \tag{1.2}$$

Where $\lambda_w=\lambda_{w_1}\cdots\lambda_{w_m}$ for the word $w=w_1\cdots w_m$. Furthermore,

$$|h(x)-h(y)|\le 2\lambda_w\cdot\|h\|_\infty,\quad \text{if}\ \ x,y\in F_wK. \tag{1.3}$$

The paper is arranged as follows. In Section 2 we show some basic facts about p.c.f. self-similar sets. In Section 3 we establish the oscillation inequality of harmonic functions and Hölder estimate of the function in the domain of the Laplacian on p.c.f. self-similar sets. In Appendix, we show that if there exists $M_{ij}=1\in M$ for a stochastic matrix $M=(M_{i,j})$, then $\delta(M)=\lambda(M)$.

2 Basic facts about p.c.f. self-similar sets

In this section we summarize some basic facts about p.c.f. self-similar sets from Kigami [4].

Let $S=\{1,2,\cdots,N\}$, and K is the unique nonempty compact set satisfying

$$K=\bigcup_{i=1}^N F_iK.$$

For $m\ge 0$, we define $\Sigma_m=S^m=\{1,2,\cdots,N\}^m$ with $\Sigma_0=\{\emptyset\}$ and call $\emptyset$ the empty word. Also, set

$$\Sigma_*=\bigcup_{m\ge 0}\Sigma_m \quad \text{and} \quad \Sigma_\infty=S^\infty.$$

Denote the length of $w\in\Sigma_*$ by $|w|$ for $w=w_1w_2\cdots w_n$. For $k\le n$, let $w|_k=w_1w_2\cdots w_k$ denote the initial segment of w of length k.