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New Fixed Point Results of Generalized g-Quasi-Contractions in Cone b-Metric Spaces Over Banach Algebras

Shaoyuan Xu[1,*], Suyu Cheng[2], Suzana Aleksić[3] and Yongjie Piao[4]

[1] *School of Mathematics and Statistics, Hanshan Normal University, Chaozhou 521041, China*
[2] *Library, Hanshan Normal University, Chaozhou 521041, China*
[3] *Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia*
[4] *Department of Mathematics, College of Science, Yanbian University, Yanji 1330022, Jilin, China*

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Abstract. In this paper, we introduce the concept of generalized g-quasi-contractions in the setting of cone b-metric spaces over Banach algebras. By omitting the assumption of normality we establish common fixed point theorems for the generalized g-quasi-contractions with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda)\in[0,\frac{1}{s})$ in the setting of cone b-metric spaces over Banach algebras, where the coefficient s satisfies $s\geq 1$. The main results generalize, extend and unify several well-known comparable results in the literature.

Key Words: Cone b-metric spaces over Banach algebras, non-normal cones, c-sequences, generalized g-quasi-contractions, fixed point theorems.

AMS Subject Classifications: 54H25, 47H10

1 Introduction

Huang and Zhang [1] introduced the concept of cone metric space, proved the properties of sequences on cone metric spaces and obtained various fixed point theorems for contractive mappings. The existence of a common fixed point on cone metric space was considered in [2–5]. Also, Ilić and Rakočević [8] introduced quasi-contraction on cone metric space when the underlying cone is normal. Later on, Kadelburg et al. [7] obtained a fixed point result without the normality of the underlying cone, but only in the case of a quasi-contractive constant $\lambda\in[0,1/2)$ (see [7, Theorem 2.2]). However, Gajić and

*Corresponding author. *Email addresses:* `xushaoyuan@126.com` (S. Xu), `chengsuyu1992@126.com` (S. Cheng), `suzanasimic@kg.ac.rs` (S. Aleksić), `sxpyj@ybu.edu.cn` (Y. Piao)

Rakočević [6] proved that result is true for $\lambda\in[0,1)$ on cone metric spaces which answered the open question whether the result is true for $\lambda\in[0,1)$. Recently, Hussain and Shah [13] introduced cone b-metric spaces, as a generalization of b-metric spaces and cone metric spaces, and established some important topological properties in such spaces. Following Hussain and Shah, Huang and Xu [10] obtained some interesting fixed point results for contractive mappings in cone b-metric spaces. Inspired by [6], Shi and Xu [31] presented a similar common fixed point result in the case of the contractive constant $\lambda\in[0,1/s)$ in cone b-metric spaces without the assumption of normality (see [31]. Similar results can be seen in [32].

Let (X,d) be a complete metric space. Recall that a mapping $T:X\to X$ is called a quasi-contraction if, for some $k\in[0,1)$ and for all $x,y\in X$, one has

$$d(Tx,Ty)\le k\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.$$

Ćirić [21] introduced and studied quasi-contractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasi-contraction T has a unique fixed point. Recently, scholars obtained various similar results on cone metric spaces. See, for instance, [6–8].

Recently, some authors investigated the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved. They used to establish the equivalence between some fixed point results in metric and in (topological vector spaces valued) cone metric spaces (see [18,19,26,27,36,37]). Very recently, Liu and Xu [22] introduced the concept of cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. Although they proved some fixed point theorems of quasi-contractions, the proof relied strongly on the assumption that the underlying cone is normal. We may state that it is significant to introduce the concept of cone metric spaces with Banach algebras (which we call in this paper cone metric spaces over Banach algebras). This is because there are examples to show that one is unable to conclude that the cone metric space (X,d) with a Banach algebra $\mathcal{A}$ discussed is equivalent to the metric space (X,d^*), where the metric d^* is defined by $d^*=\xi_e\circ d$, here the nonlinear scalarization function $\xi_e:A\to\mathbb{R}$ $(e\in\mathrm{int}P)$ is defined by

$$\xi_e(y)=\inf\{r\in\mathbb{R}:y\in re-P\}.$$

See [18,19,22,26–28] for more details.

In the present paper we introduce the concept of generalized g-quasi-contractions in cone b-metric spaces over Banach algebras and obtain common fixed point theorems for two weakly compatible self-mappings satisfying the g-quasi-contractive condition in the case of the g-quasi-contractive constant vector with $r(\lambda)\in[0,1/s)$ in cone b-metric spaces without the assumption of normality, where the coefficient s satisfies $s\ge 1$. As consequences, our main results not only extend the fixed point theorem of g-quasi-contractions in cone b-metric spaces to the case in cone b-metric spaces over Banach algebras, but also