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Regularized Interpolation Driven by Total Variation

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Abstract. We explore minimization problems of the form

$$\operatorname{Inf}\left\{\int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \alpha \int_0^1 |u|^2\right\},$$

where u is a function defined on $(0,1)$, (a_i) are k given points in $(0,1)$, with $k \geq 2$, (f_i) are k given real numbers, and $\alpha \geq 0$ is a parameter taken to be 0 or 1 for simplicity. The natural functional setting is the Sobolev space $W^{1,1}(0,1)$. When $\alpha = 0$ the Inf is achieved in $W^{1,1}(0,1)$. However, when $\alpha = 1$, minimizers need not exist in $W^{1,1}(0,1)$. One is led to introduce a relaxed functional defined on the space $BV(0,1)$, whose minimizers always exist and can be viewed as generalized solutions of the original ill-posed problem.

Key Words: Interpolation, minimization problems, functions of bounded variation, relaxed functional.

AMS Subject Classifications: 26B30, 49J45, 65D05

1 Introduction

Given k points, with $k \geq 2$,

$$0 < a_1 < a_2 < \cdots < a_k < 1, \tag{1.1}$$

and k real numbers f_i, $i = 1, \cdots, k$, the aim is to find a function u defined on $(0,1)$ such that $u(a_i)$ approximates f_i as best as possible, and keeping at the same time some control on the regularity of u, measured here in terms of total variation of u. For this purpose define the functional

$$F(u) = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2, \tag{1.2}$$

*Corresponding author. *Email address:* `brezis@math.rutgers.edu` (H. Brezis) and then minimize F. (One may also insert a fidelity parameter in front of the first integral, but we take to be 1 for simplicity). Note that F is well-defined on the Sobolev space $W^{1,1}(0,1)$ since $W^{1,1}(0,1) \subset C([0,1])$, so that $u(a_i)$ makes sense. As is well-known $W^{1,1}(0,1)$ is not a good function space from the point of view of minimization techniques in Functional Analysis. Often, variational problems do *not* admit minimizers in $W^{1,1}(0,1)$. To make up for this "defect" one is usually led to enlarge $W^{1,1}(0,1)$ and replace it by $BV(0,1)$, the space of functions of bounded variation (see e.g., [1,2,5]), where the existence of minimizers is often a matter of routine. The drawback is that the specific functional F is not properly defined on $BV(0,1)$ since the term $u(a_i)$ has no obvious meaning when u has a jump at a_i.

In Section 2 we establish that (surprisingly!) the problem

$$\operatorname*{Inf}_{u\in W^{1,1}(0,1)} F(u) \tag{1.3}$$

always admits minimizers. In fact all minimizers are classified with the help of a finite-dimensional auxiliary problem. Given

$$\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k,$$

set

$$\Phi(\lambda) := \sum_{i=1}^{k-1} |\lambda_{i+1} - \lambda_i| + \sum_{i=1}^{k} |\lambda_i - f_i|^2. \tag{1.4}$$

By convexity

$$m := \min_{\lambda\in\mathbb{R}^k} \Phi(\lambda) \tag{1.5}$$

is achieved by some unique λ denoted

$$U = (U_1, \cdots, U_k),$$

and which plays an important role throughout the paper. In this section we never invoke Functional Analysis and the space $BV(0,1)$ is noticeably absent. The existence of minimizers in $W^{1,1}(0,1)$ is derived from an *elementary* computation originally due to T. Sznigir [6,7]. However this "miracle" does not repeat itself: as we are going to see in Section 5 even "mild" pertubations of F need not admit minimizers in $W^{1,1}(0,1)$, and there it will be essential to "relax" the problem and search for minimizers in $BV(0,1)$ using tools of Functional Analysis.

In Section 3 we introduce the relaxed functional F_r of F, which is much better suited to minimization problems involving the functional F. We start with the standard abstract formulation, namely F_r is defined for every $v \in BV(0,1)$ by

$$F_r(v) := \operatorname{Inf} \liminf_{n\to\infty} F(v_n), \tag{1.6}$$

where the Inf in (1.6) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. The main result, Theorem 3.1, provides an *explicit* formula for F_r. The major obstacle stems from the fact that $u(a)$ is not well-defined when $u \in BV(0,1)$; however, u admits at every point $a \in (0,1)$ limits from the left and from the right, which enter in the formula for F_r. Theorem 4.1 provides a complete description of all minimizers of F_r on $BV(0,1)$. It turns out that F_r admits *many more* minimizers than the original functional F, even when F_r is restricted to $W^{1,1}(0,1)$.

In Section 5 we consider a mild perturbation of F, and we show that the corresponding minimizing problems differ significantly from those associated with F. Set

$$G(u) = F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \int_0^1 |u|^2, \tag{1.7}$$

where $u \in W^{1,1}(0,1)$. Our initial goal is to investigate the minimization problem

$$A = \operatorname*{Inf}_{u \in W^{1,1}} G(u). \tag{1.8}$$

As we are going to see the infimum in (1.8) need *not* be achieved and we will replace it by a relaxed problem defined on $BV(0,1)$ as we have done in Section 3. It is easy to check that the relaxed functional G_r of G is given by

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \qquad \forall v \in BV(0,1), \tag{1.9}$$

so that G_r is strictly convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n \to \infty} G_r(v_n) \geq G_r(v).$$

Consequently

$$B = \operatorname*{Inf}_{v \in BV} G_r(v) \tag{1.10}$$

is uniquely achieved and we denote by $\bar{v} \in BV(0,1)$, its unique minimizer.

The bottom line is that we have replaced Problem (1.8) which need *not* have a solution by Problem (1.10) which *always* admits a unique solution $\bar{v}$. Moreover, *if* Problem (1.8) admits a minimizer, it must coincide with $\bar{v}$. Therefore $\bar{v}$ may be viewed as *the generalized solution* of Problem (1.8). In addition, $\bar{v}$ has a very simple structure and can be computed via a *finite-dimensional* convex minimization problem.

2 The functional F and its minimizers on $W^{1,1}$

The main result in this section is

Theorem 2.1 (T. Sznigir [6,7]). *We have*

$$m = \operatorname*{Inf}_{u \in W^{1,1}} F(u), \tag{2.1}$$

where m has been defined in (1.5), *and the* Inf *in* (2.1) *is achieved. More precisely* $u \in W^{1,1}(0,1)$ *is a minimizer if and only if it satisfies the following three conditions:*

$$u \text{ is monotone on each interval } (a_i, a_{i+1}), \quad i = 1, \cdots, k-1, \tag{2.2a}$$

$$u(a_i) = U_i, \quad i = 1, \cdots k, \tag{2.2b}$$

$$u(x) = U_1, \quad \forall x \in [0, a_1] \quad \text{and} \quad u(x) = U_k, \quad \forall x \in [a_k, 1]. \tag{2.2c}$$

Proof. Given $u \in W^{1,1}(0,1)$ we have

$$\int_0^1 |u'| \geq \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |u'| \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)|, \tag{2.3}$$

with equalities if and only if:

$$u \text{ is monotone on each interval } \ (a_i, a_{i+1}), \tag{2.4a}$$

$$u \text{ is constant on } (0, a_1) \text{ and on } \ (a_k, 1). \tag{2.4b}$$

Thus

$$F(u) \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)| + \sum_{i=1}^{k} |u(a_i) - f_i|^2.$$

Letting $\lambda_i = u(a_i)$, $i = 1, \cdots, k$, we see that, for every $u \in W^{1,1}(0,1)$,

$$F(u) \geq \min_{\lambda \in \mathbb{R}^k} \Phi(\lambda) = m. \tag{2.5}$$

If $u \in W^{1,1}(0,1)$ satisfies (2.2a)-(2.2c) we have

$$F(u) = \sum_{i=1}^{k-1} |U_{i+1} - U_i| + \sum_{i=1}^{k} |U_i - f_i|^2 = m,$$

so that u is a minimizer for (2.1). Conversely if $u \in W^{1,1}(0,1)$ is such that $F(u) = m$ then (2.4a) and (2.4b) hold. Moreover $u(a_i) = \lambda_i$ is a minimizer in (1.5), and by uniqueness we have $u(a_i) = U_i$ for $i = 1, \cdots, k$. □

Remark 2.1. In view of the abundance of minimizers for F in $W^{1,1}(0,1)$ one may wonder whether some of them are "preferred" e.g., in the sense that they are "stable" with respect to pertubations. The minimizer u_ℓ of F which is obtained by *linear* interpolation (i.e., u_ℓ is linear on each interval (a_i, a_{i+1}) is definitely a good candidate. Here are three "natural" perturbed functionals:

$$F_{1,\varepsilon}(u) = \varepsilon \int_0^1 |u'|^2 + F(u), \qquad u \in H^1(0,1), \qquad \varepsilon > 0,$$

$$F_{2,p}(u) = \int_0^1 |u'|^p + \sum_{i=1}^{k} |u(a_i) - f_i|^2, \qquad u \in W^{1,p}(0,1), \qquad p > 1,$$

$$F_{3,\varepsilon}(u) = \varepsilon \int_0^1 |u''|^2 + F(u), \qquad u \in H^2(0,1), \qquad \varepsilon > 0.$$

It is easy to see that each one admits a unique minimizer. T. Sznigir [6,7] has established that as $\varepsilon \to 0$ (resp. $p \searrow 1$) the minimizers of $F_{1,\varepsilon}$ (resp. $F_{2,p}$) converge to u_ℓ. By contrast the minimizers of $F_{3,\varepsilon}$ converge as $\varepsilon \to 0$ to the solution $\hat{u}$ of a variational inequality corresponding to

$$\min\left\{\int_0^1 |u''|^2;\ u \in H^2(0,1) \ \text{ and satisfies } \ \text{(2.2a)} - \text{(2.2c)}\right\}.$$

The function $\hat{u}$ belongs to $C^1([0,1])$ (while $u_\ell \notin C^1$) and $\hat{u}$ is a piecewise cubic function on each interval (a_i, a_{i+1}), $i = 1, \cdots, k-1$, see [6,7].

3 The relaxed functional F_r on BV

As usual the relaxed functional F_r is defined for every $v \in BV(0,1)$ by

$$F_r(v) := \ \text{Inf} \ \liminf_{n\to\infty} F(v_n), \tag{3.1}$$

where the Inf in (3.1) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ (the choice of L^2 is just a matter of convenience–one can replace it by any L^p, $1 \le p < \infty$).

The main result in this section is an *explicit* formula for F_r, but first some notation. Given $v \in BV(0,1)$ and $a \in (0,1)$ we denote by $j(v)(a)$ the jump interval of v at a, i.e.,

$$j(v)(a) = [\min(v(a-0), v(a+0)), \max(v(a-0), v(a+0))]. \tag{3.2}$$

We also set

$$\varphi(t) = \begin{cases} t^2, & \text{if } \ 0 \le t \le 1, \\ 2t-1, & \text{if } \ t > 1. \end{cases} \tag{3.3}$$

Theorem 3.1. *For every $v \in BV(0,1)$, we have*

$$F_r(v) = \int_0^1 |v'| + \sum_{i=1}^k \varphi\left(\text{dist}(f_i, j(v)(a_i))\right), \tag{3.4}$$

where dist *denotes the distance of a point to a set.*

The proof of Theorem 3.1 relies on the following three lemmas. The first two are familiar to the experts (see e.g., [4, Appendix 18.8] and [3, Lemma 2]).

Lemma 3.1. *Let (v_n) be a bounded sequence in $BV(a,b)$ such that $v_n \to v$ in $L^1(a,b)$, $v_n(a) \to \alpha$, $v_n(b) \to \beta$ as $n \to \infty$. Then $v \in BV(a,b)$ and*

$$\liminf_{n\to\infty} \int_a^b |v_n'| \ge \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|, \tag{3.5}$$

where we write for simplicity $v_n(a) = v_n(a+0)$, etc.

Proof. Fix any function $h \in C_c^\infty(\mathbb{R})$ such that $h(a) = \alpha$ and $h(b) = \beta$. Consider the functions

$$w_n(t) := \begin{cases} h(t), & \text{if } t < a, \\ v_n(t), & \text{if } a \le t \le b, \\ h(t), & \text{if } t > b, \end{cases} \qquad w(t) := \begin{cases} h(t), & \text{if } t < a, \\ v(t), & \text{if } a \le t \le b, \\ h(t), & \text{if } t > b. \end{cases}$$

Clearly $w_n, w \in BV(\mathbb{R})$ and

$$\int_{\mathbb{R}} |w_n'| = \int_{-\infty}^{a} |h'| + \int_a^b |v_n'| + \int_b^\infty |h'| + |v_n(a) - \alpha| + |v_n(b) - \beta|, \tag{3.6a}$$

$$\int_{\mathbb{R}} |w'| = \int_{-\infty}^{a} |h'| + \int_a^b |v'| + \int_b^\infty |h'| + |v(0) - \alpha| + |v(b) - \beta|. \tag{3.6b}$$

Since $w_n \to w$ in $L^1(\mathbb{R})$ it is well-known that

$$\liminf_{n\to\infty} \int_{\mathbb{R}} |w_n'| \ge \int_{\mathbb{R}} |w'|.$$

Combining this with (3.6) yields (3.5). □

Lemma 3.2. *Given any $v \in BV(a,b)$ and constants $\alpha, \beta \in \mathbb{R}$, there exists a sequence $(v_n) \subset W^{1,1}(a,b)$ such that $v_n \to v$ in $L^2(a,b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and*

$$\lim_{n\to\infty} \int_a^b |v_n'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{3.7}$$

Proof. Set

$$w(t) := \begin{cases} \alpha, & \text{if } t < a, \\ v(t), & \text{if } a \le t \le b, \\ \beta, & \text{if } t > b. \end{cases}$$

Let $w_n = \rho_n * w$ where (ρ_n) is a sequence of mollifiers. Clearly

$$\int_{\mathbb{R}} |w_n'| \le \int_{\mathbb{R}} |w'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{3.8}$$

Moreover $w_n(t) = \alpha$ if $t < a - (1/n)$ and $w_n(t) = \beta$ if $t > b + (1/n)$. Rescaling the sequence (w_n) by a change of variables we obtain a sequence (v_n) of smooth functions such that $v_n \to v$ in $L^2(a,b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and

$$\limsup_{n\to\infty} \int_a^b |v_n'| \le \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|.$$

Applying Lemma 3.1 we conclude that (3.7) holds. □

The third lemma relies on an elementary computation left to the reader.

Lemma 3.3. *Given any $\alpha,\beta,f\in\mathbb{R}$ we have*

$$\operatorname*{Inf}_{t\in\mathbb{R}}\left\{|t-\alpha|+|t-\beta|+|t-f|^2\right\}=|\alpha-\beta|+\varphi(\operatorname{dist}(f,J)), \tag{3.9}$$

where $J=[\min(\alpha,\beta),\max(\alpha,\beta)]$ and φ has been defined in (3.1).

Proof of Theorem 3.1. It consists of two steps.

Step 1. Given any $v\in BV(0,1)$ there exists a sequence $(v_n)\subset W^{1,1}(0,1)$ such that $v_n\to v$ in $L^2(0,1)$ and

$$\lim_{n\to\infty}F(v_n)=F_r(v),$$

where $F_r(v)$ is defined by (3.4).

Proof. Applying Lemma 3.3 with $\alpha=v(a_i-0)$, $\beta=v(a_i+0)$, and $f=f_i$, $1\le i\le k$, we obtain some t_i (a minimizer in (3.9)) such that

$$\begin{aligned}&|t_i-v(a_i-0)|+|t_i-v(a_i+0)|+|t_i-f_i|^2\\=&|v(a_i-0)-v(a_i+0)|+\varphi(\operatorname{dist}(f_i,j(v)(a_i))).\end{aligned} \tag{3.10}$$

We next apply Lemma 3.2 successively on

$$(0,a_1),\quad (a_i,a_{i+1}),\quad 1\le i\le k-1,\quad \text{and}\quad (a_k,1).$$

First on $(0,a_1)$ with $\alpha=v(0+)$ and $\beta=t_1$. This yields a sequence $(v_n)\subset W^{1,1}(0,a_1)$ such that $v_n(0)=v(0+)$, $v_n(a_1)=t_1$, $\forall n$, $v_n\to v$ in $L^2(0,a_1)$, and

$$\int_0^{a_1}|v_n'|=\int_0^{a_1}|v'|+|v(a_1-0)-t_1|+o(1). \tag{3.11}$$

Next on (a_i,a_{i+1}), $1\le i\le k-1$, with $\alpha=t_i$ and $\beta=t_{i+1}$; this yields a sequence $(v_n)\subset W^{1,1}(a_i,a_{i+1})$ such that $v_n(a_i)=t_i$, $v_n(a_{i+1})=t_{i+1}$, $v_n\to v$ in $L^2(a_i,a_{i+1})$, and

$$\int_{a_i}^{a_{i+1}}|v_n'|=\int_{a_i}^{a_{i+1}}|v'|+|v(a_i+0)-t_i|+|v(a_{i+1}-0)-t_{i+1}|+o(1). \tag{3.12}$$

Finally on $(a_k,1)$ with $\alpha=t_k$ and $\beta=v(1-)$; this yields a sequence $(v_n)\subset W^{1,1}(a_k,1)$ such that $v_n(a_k)=t_k$, $v_n(1)=v(1-)$, $v_n\to v$ in $L^2(a_k,1)$, and

$$\int_{a_k}^{1}|v_n'|=\int_{a_k}^{1}|v'|+|v(a_k+0)-t_k|+o(1). \tag{3.13}$$

Glueing these functions we obtain a sequence $(v_n)\subset W^{1,1}(0,1)$ such that $v_n\to v$ in $L^2(0,1)$, and

$$\int_0^1|v_n'|=\sum_{i=0}^{k}\int_{a_i}^{a_{i+1}}|v'|+\sum_{i=1}^{k}(|v(a_i-0)-t_i|+|v(a_i+0)-t_i|)+o(1), \tag{3.14}$$

with the convention that $a_0=0$ and $a_{k+1}=1$.

Inserting (3.10) into (3.14) we see that

$$\int_0^1 |v_n'| = \int_0^1 |v'| - \sum_{i=1}^k |t_i - f_i|^2 + \sum_{i=1}^k \varphi(\text{dist }(f_i, j(v)(a_i))) + o(1). \tag{3.15}$$

Since $v_n(a_i)=t_i$, $\forall n, \forall i$, we conclude that

$$F(v_n) = \int_0^1 |v_n'| + \sum_{i=1}^k |v_n(a_i) - f_i|^2 = F_r(v) + o(1), \tag{3.16}$$

which completes the proof of Step 1.

Step 2. Let (v_n) be a bounded sequence in $W^{1,1}(0,1)$ such that $v_n \to v$ in $L^1(0,1)$. Then $v \in BV(0,1)$ and

$$\liminf_{n\to\infty} F(v_n) \geq F_r(v). \tag{3.17}$$

Proof. Passing to a subsequence we may always assume that, for every $i=0,1,\cdots,k+1$, there exists some ℓ_i such that

$$v_n(a_i) \to \ell_i \quad \text{as} \quad n\to\infty.$$

From Lemma 3.1 we know that for every $i=0,1,\cdots k$,

$$\int_{a_i}^{a_{i+1}} |v_n'| \geq \int_{a_i}^{a_{i+1}} |v'| + |v(a_i+0) - \ell_i| + |v(a_{i+1}-0) - \ell_{i+1}| + o(1).$$

Adding these inequalities yields

$$F(v_n) \geq \sum_{i=0}^k \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^k (|v(a_i+0) - \ell_i| + |v(a_i-0) - \ell_i| + |\ell_i - f_i|^2) + o(1).$$

Applying Lemma 3.3 we find that

$$\begin{aligned} F(v_n) \geq &\sum_{i=0}^k \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^k |v(a_i+0) - v(a_i-0)| + \sum_{i=1}^k \varphi(\text{dist }(f_i, j(v)(a_i)) + o(1) \\ =&F_r(v) + o(1), \end{aligned}$$

which completes the proof of Step 2, and thereby the proof of Theorem 3.1. □

4 Some properties of F_r

We discuss in this section some properties of F_r. First a few straightforward facts. We have

$$F_r(v) \leq F(v), \qquad \forall v \in W^{1,1}(0,1), \tag{4.1}$$

indeed it suffices to choose $v_n = v, \forall n$ in (3.1). It may happen that $F_r(v) < F(v)$ for some $v's$ in $W^{1,1}(0,1)$. In fact

$$[F_r(v) = F(v) \quad \text{for some} \quad v \in W^{1,1}(0,1)] \Leftrightarrow [|v(a_i) - f_i| \leq 1, \ \ \forall i = 1, \cdots, k], \tag{4.2}$$

this is an immediate consequence of (1.2), (3.4) and (3.3).

Lemma 4.1. *The functional F_r is convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have*

$$\liminf_{n\to\infty} F_r(v_n) \geq F_r(v). \tag{4.3}$$

Proof. Given $v, w \in BV(0,1)$ there exist (by Step 1 above) sequences $(v_n), (w_n) \subset W^{1,1}(0,1)$ such that $v_n \to v, w_n \to w$ in $L^2(0,1)$ and $F(v_n) \to F_r(v), F(w_n) \to F_r(w)$. By convexity of F we have

$$F(tv_n + (1-t)w_n) \leq tF(v_n) + (1-t)F(w_n), \qquad \forall t \in [0,1]. \tag{4.4}$$

Passing to the limit in (4.4) and using Step 2 we see that

$$F_r(tv + (1-t)w) \leq tF_r(v) + (1-t)F_r(w).$$

Next, the proof of (4.3). By Step 1 applied to v_n with n fixed we may find some $w_n \in W^{1,1}(0,1)$ such that

$$\|v_n - w_n\|_{L^2} < \frac{1}{n} \quad \text{and} \quad |F_r(v_n) - F(w_n)| < \frac{1}{n}. \tag{4.5}$$

Thus $w_n \to v$ in $L^2(0,1)$ and from the definition (3.1) we conclude that

$$F_r(v) \leq \liminf_{n\to\infty} F(w_n) = \liminf_{n\to\infty} F_r(v_n) \quad \text{by} \quad (4.5).$$

Thus, we complete the proof. □

We now discuss the minization of F_r on $BV(0,1)$. Recall (see Theorem 2.1) that

$$m = \min_{v \in W^{1,1}} F(v), \tag{4.6}$$

where m is defined by (1.5). Set

$$\mu := \operatorname*{Inf}_{v \in BV} F_r(v). \tag{4.7}$$

From Lemma 4.1 and the compactness of the embedding $BV(0,1) \subset L^2(0,1)$ we deduce that the Inf in (4.7) is achieved. Clearly, by (4.1),

$$\mu = \operatorname*{Inf}_{v \in BV} F_r(v) \leq \operatorname*{Inf}_{v \in W^{1,1}} F_r(v) \leq \operatorname*{Inf}_{v \in W^{1,1}} F(v) = m. \tag{4.8}$$

We claim that

$$\mu = m. \tag{4.9}$$

Indeed, by (4.6) we have

$$m \le F(v), \qquad \forall v \in W^{1,1}(0,1). \tag{4.10}$$

From Step 1 above and (4.10) we deduce that

$$m \le F_r(v), \qquad \forall v \in BV(0,1), \tag{4.11}$$

and thus

$$m \le \operatorname*{Inf}_{v\in BV} F_r(v) = \mu.$$

Combined with (4.8) this yields (4.9).

As a consequence, any minimizer for F on $W^{1,1}(0,1)$ must be a minimizer for F_r on $BV(0,1)$. Indeed if $F(u) = m$, then $\mu \le F_r(u) \le F(u) = m = \mu$ so that $F_r(u) = \mu$.

The next result provides a complete description of all minimizers of F_r on $BV(0,1)$.

Theorem 4.1. *Assume that $u \in BV(0,1)$ satisfies the following three conditions:*

$$\begin{cases} u \text{ is monotone nondecreasing (resp. nonincreasing)} \\ \text{on each interval } (a_i, a_{i+1}),\ i = 1,\cdots,k-1, \\ \text{such that } U_i \le U_{i+1} (\text{ resp. } U_{i+1} \le U_i), \end{cases} \tag{4.12a}$$

$$\begin{cases} U_i \le u(a_i+0) \ \text{ and } \ u(a_{i+1}-0) \le U_{i+1} \ \text{ if} \\ U_i \le U_{i+1} (\text{resp. reverse inequalities if } \ U_{i+1} \le U_i), \end{cases} \tag{4.12b}$$

$$u(x) = U_1, \quad \forall x \in [0,a_1] \quad \text{and} \quad u(x) = U_k, \quad \forall x \in [a_k, 1], \tag{4.12c}$$

then u is a minimizer of F_r on $BV(0,1)$. And conversely.

Remark 4.1. We deduce from Theorem 4.1 that the relaxed functional F_r admits *many more* minimizers than the original functional F, even when F_r is restricted to $W^{1,1}(0,1)$, since they are not bound by the rigid constraint $u(a_i) = U_i$, $\forall i$.

The proof relies on the following monotone version of Lemma 3.2.

Lemma 4.2. *Given any nondecreasing function v on (a,b) and constants $\alpha \le v(a)$, $\beta \ge v(b)$, there exists a sequence of nondecreasing functions $(v_n) \subset W^{1,1}(a,b)$ such that $v_n \to v$ in $L^2(a,b), v_n(a) = \alpha, v_n(b) = \beta \quad \forall n$, and*

$$\lim_{n\to\infty} \int_b^a |v_n'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{4.13}$$

The proof of Lemma 4.2 which is similar to the proof of Lemma 3.2 is left to the reader. We now turn to the

Proof of Theorem 4.1. Applying Lemma 4.2 on the interval (a_i, a_{i+1}) with $\alpha = U_i$ and $\beta = U_{i+1}$ we obtain a sequence (v_n) of monotone functions in $W^{1,1}(a_i, a_{i+1})$ such that $v_n \to u$ in $L^2(a_i, a_{i+1})$,

$$v_n(a_i) = U_i \quad \text{and} \quad v_n(a_{i+1}) = U_{i+1}, \qquad \forall n, \tag{4.14a}$$

$$\lim_{n\to\infty} \int_{a_i}^{a_{i+1}} |v_n'| = \int_{a_i}^{a_{i+1}} |u'| + |u(a_i+0) - U_i| + |u(a_{i+1}-0) - U_{i+1}|. \tag{4.14b}$$

Next we set

$$v_n(x) = U_1, \quad \forall x \in [0, a_1] \quad \text{and} \quad v_n(x) = U_k, \quad \forall x \in [a_k, 1]. \tag{4.15}$$

Glueing the functions v_n defined above we obtain a function still denoted $v_n \in W^{1,1}(0,1)$, satisfying all the requirements of Theorem 2.1. Thus v_n is a minimizer for F in $W^{1,1}(0,1)$ so that

$$F(v_n) = m, \qquad \forall n. \tag{4.16}$$

Since $v_n \to u$ in $L^2(0,1)$, we deduce (from the definition (3.1) of F_r) that

$$F_r(u) \le \lim_{n\to\infty} F(v_n) = m.$$

(Note that the full strength of Lemma 4.1 was not used). Invoking (4.9) we conclude that u is a minimizer for F_r.

We now turn to the converse. Assume that u is a minimizer for F_r on $BV(0,1)$. Let t_i, $i = 1, \cdots, k$, be the unique minimizer in (3.9) corresponding to $\alpha = u(a_i - 0)$, $\beta = u(a_i + 0)$ and $f = f_i$, so that

$$\begin{aligned} &|t_i - u(a_i - 0)| + |t_i - u(a_i + 0)| + |t_i - f_i|^2 \\ =&|u(a_i - 0) - u(a_i + 0)| + \varphi(\mathrm{dist}(f_i, j(u)(a_i))). \end{aligned} \tag{4.17}$$

Next write, for $1 \le i \le k-1$,

$$t_i - t_{i+1} = (t_i - u(a_i+0)) + (u(a_i+0) - u(a_{i+1}-0)) + (u(a_{i+1}-0) - t_{i+1}), \tag{4.18}$$

so that

$$|t_i - t_{i+1}| \le |t_i - u(a_i+0)| + |u(a_i+0) - u(a_{i+1}-0)| + |u(a_{i+1}-0) - t_{i+1}|. \tag{4.19}$$

We now compute, as in (1.4),

$$\Phi(\bar{t}) = \sum_{i=1}^{k-1} |t_i - t_{i+1}| + \sum_{i=1}^{k} |t_i - f_i|^2,$$

where $\bar{t}$ is defined by $\bar{t} := (t_1, \cdots, t_k)$. From (4.19) and (4.17) we have when $k \geq 3$ (if $k = 2$ go directly to (4.20))

$$
\begin{aligned}
&\sum_{i=1}^{k-1} |t_i - t_{i+1}| \\
\leq &\sum_{i=1}^{k-1} |t_i - u(a_i + 0)| + \sum_{i=2}^{k} |t_i - u(a_i - 0)| + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)| \\
= &|t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + \sum_{i=2}^{k-1} (|t_i - u(a_i - 0)| + |t_i - u(a_i + 0)|) \\
&\quad + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)| \\
= &|t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + \sum_{i=2}^{k-1} |u(a_i + 0) - u(a_i - 0)| \\
&\quad + \sum_{i=2}^{k-1} \varphi(\mathrm{dist}(f_i, j(u)(a_i)) - \sum_{i=2}^{k-1} |t_i - f_i|^2 + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Phi(\bar{t}) = &\sum_{i=1}^{k-1} |t_i - t_{i+1}| + \sum_{i=1}^{k} |t_i - f_i|^2 \\
\leq &|t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 + \sum_{i=2}^{k-1} |u(a_i + 0) \\
&\quad - u(a_i - 0)| + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)| + \sum_{i=2}^{k-1} \varphi(\mathrm{dist}(f_i, j(u)(a_i)) \\
\leq &|t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 \\
&\quad + \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |u'| + \sum_{i=2}^{k-1} |u(a_i + 0) - u(a_i - 0)| + \sum_{i=2}^{k-1} \varphi(\mathrm{dist}\ (f_i, j(u)(a_i)) \\
= &|t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 \\
&\quad + \int_0^1 |u'| - \int_0^{a_1} |u'| - \int_{a_k}^1 |u'| - |u(a_1 + 0) - u(a_1 - 0)| \\
&\quad - |u(a_k + 0) - u(a_k - 0)| + \sum_{i=2}^{k-1} \varphi(\mathrm{dist}(f_i, j(u)(a_i)).
\end{aligned}
$$

Since u is a minimizer for F_r we know by (4.7) and (4.9) that

$$
F_r(u) = \int_0^1 |u'| + \sum_{i=1}^{k} \varphi(\mathrm{dist}(f_i, j(u)(a_i)) = m,
$$

so that

$$\begin{aligned}\Phi(\bar{t}) \leq &|t_1 - u(a_1+0)| + |t_k - u(a_k-0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 \\ &+ m - \varphi(\operatorname{dist}(f_1, j(u)(a_1)) - \varphi(\operatorname{dist}(f_k, j(u)(a_k)) - \int_0^{a_1} |u'| \\ &- \int_{a_k}^1 |u'| - |u(a_1+0) - u(a_1-0)| - |u(a_k+0) - u(a_k-0)|. \end{aligned} \tag{4.20}$$

Finally we use (4.17) for $i = 1$ and $i = k$, and deduce from (4.20) that

$$\Phi(\bar{t}) \leq -|t_1 - u(a_1-0)| - |t_k - u(a_k+0)| - \int_0^{a_1} |u'| - \int_{a_k}^1 |u'| + m. \tag{4.21}$$

Therefore

$$\Phi(\bar{t}) \leq m,$$

so that by (1.5), $\bar{t} = (t_1, \cdots, t_k)$ is a minimizer of Φ on $\mathbb{R}^k$. By uniqueness we have

$$t_i = U_i, \qquad \forall i. \tag{4.22}$$

Moreover from (4.21) we deduce that

$$|t_1 - u(a_1-0)| = |t_k - u(a_k+0)| = \int_0^{a_1} |u'| = \int_0^{a_k} |u'| = 0.$$

Consequently (4.12c) holds. Returning to the above estimates we infer that all inequalities are equalities. In particular, $\forall i = 1, \cdots, k-1$,

$$\int_{a_i}^{a_{i+1}} |u'| = |u(a_i+0) - u(a_{i+1}-0)| \tag{4.23}$$

and

$$|t_i - t_{i+1}| = |t_i - u(a_i+0)| + |u(a_i+0) - u(a_{i+1}-0)| + |u(a_{i+1}-0) - t_{i+1}|. \tag{4.24}$$

Equality (4.23) implies that u is monotone on the interval (a_i, a_{i+1}), while equality (4.24) yields

$$\begin{aligned}\operatorname{sign}(t_i - t_{i+1}) =&\operatorname{sign}(t_i - u(a_i+0)) = \operatorname{sign}(u(a_i+0) - u(a_{i+1}-0)) \\ =&\operatorname{sign}(u(a_{i+1}-0) - t_{i+1}).\end{aligned}$$

In view of (4.22) we conclude easily that u satisfies (4.12a) - (4.12b). $\square$

Remark 4.2. Theorem 4.1 is stated in T. Sznigir [6] though the proof in [6] is somewhat obscure.

Remark 4.3. We have

$$|U_i - f_i| \leq 1, \qquad \forall i = 1, \cdots, k. \tag{4.25}$$

Indeed consider the piecewise linear function u_ℓ defined in Remark 2.1. Then u_ℓ satisfies

$$m = F(u_\ell) = \mu = F_r(u_\ell).$$

In view of (4.2) this implies (4.25). Inequality (4.25) could also be deduced directly from the fact that $U = (U_1, \cdots, U_k)$ is a minimizer of Φ defined in (1.4). We have, using the theory of sub-differentials,

$$0 \in 2(U_i - f_i) + \text{ Sign } (U_i - U_{i-1}) + \text{ Sign } (U_i - U_{i+1}), \qquad \forall i = 2, \cdots, k-1, \tag{4.26}$$

where Sign denotes as usual the monotone graph defined by

$$\text{Sign}(s) := \begin{cases} +1, & \text{if } \ s > 0, \\ [-1, +1], & \text{if } \ s = 0, \\ -1, & \text{if } \ s < 0. \end{cases}$$

This implies (4.25). On the other hand, we have

$$0 \in 2(U_1 - f_1) + \text{ Sign } (U_1 - U_2),$$

and

$$0 \in 2(U_k - f_k) + \text{ Sign } (U_k - U_{k-1}),$$

which imply in fact that

$$|U_1 - f_1| \leq \frac{1}{2} \text{ and } |U_k - f_k| \leq \frac{1}{2}.$$

5 Where a mild pertubation can produce a big difference

In this section we consider a mild pertubation of the original functional F defined by (1.2) and we show that the corresponding minimizing problems differ significantly from those associated with F.

Set

$$G(u) = F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2 + \int_0^1 |u|^2, \tag{5.1}$$

where $u \in W^{1,1}(0,1)$. Our initial goal is to investigate the minimization problem

$$A = \operatorname*{Inf}_{u \in W^{1,1}} G(u). \tag{5.2}$$

It turns out that the infimum in (5.2) need *not* be achieved (see [6, 7] and Remark 5.2) and we will replace it by a relaxed problem defined on $BV(0,1)$ as we have done in Section 3. For every $v \in BV(0,1)$ set

$$G_r(v) = \operatorname{Inf} \liminf_{n\to\infty} G(v_n), \tag{5.3}$$

where the Inf in (5.3) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. It is easy to check that

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \qquad \forall v \in BV(0,1), \tag{5.4}$$

so that G_r is strictly convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} G_r(v_n) \geq G_r(v).$$

Consequently

$$B = \operatorname*{Inf}_{v\in BV} G_r(v) \tag{5.5}$$

is uniquely achieved, and we denote by $\bar{v} \in BV(0,1)$ its unique minimizer, i.e.,

$$B = G_r(\bar{v}). \tag{5.6}$$

We claim that

$$A = B. \tag{5.7}$$

From (4.1), we deduce that $G_r \leq G$ on $W^{1,1}(0,1)$, and thus

$$B = \operatorname*{Inf}_{v\in BV} G_r(v) \leq \operatorname*{Inf}_{v\in W^{1,1}} G_r(v) \leq \operatorname*{Inf}_{v\in W^{1,1}} G(v) = A. \tag{5.8}$$

On the other hand we have by (5.2)

$$A \leq G(u) = F(u) + \int_0^1 |u|^2, \qquad \forall u \in W^{1,1}(0,1). \tag{5.9}$$

From (5.9) and Step 1 in Section 3 we deduce that

$$A \leq F_r(v) + \int_0^1 |v|^2 = G_r(v), \qquad \forall v \in BV(0,1), \tag{5.10}$$

and thus

$$A \leq \operatorname*{Inf}_{v\in BV} G_r(v) = B. \tag{5.11}$$

Combining (5.11) with (5.8) yields $A = B$.

As a consequence, if Problem (5.2) admits a minimizer $v_0 \in W^{1,1}(0,1)$, then

$$B \le G_r(v_0) \le G(v_0) = A,$$

so that, by (5.7), $G_r(v_0) = B$, i.e., v_0 is a minimizer for Problem (5.5). By uniqueness $v_0 = \bar{v}$.

The bottom line is that we have replaced Problem (5.2) which need *not* have a solution by Problem (5.5) which *always* admits a unique solution $\bar{v}$. Therefore $\bar{v}$ may be viewed as *the generalized solution* of Problem (5.2).

Remark 5.1. This concept of generalized solution is quite robust. In particular if $(u_n) \subset W^{1,1}(0,1)$ is a minimizing sequence for (5.2), then $u_n \to \bar{v}$ as $n \to \infty$ in $L^2(0,1)$. Indeed, we have

$$G_r(u_n) \le G(u_n) \le A + o(1),$$

and a subsequence (u_{n_k}) converges in $L^2(0,1)$ to some $\bar{u} \in BV(0,1)$ satisfying

$$B \le G_r(\bar{u}) = A,$$

so that $G_r(\bar{u}) = B$ and by uniqueness $\bar{u} = \bar{v}$. Similarly, if we consider as in Remark 2.1,

$$G_{1,\varepsilon}(u) = F_{1,\varepsilon}(u) + \int_0^1 |u|^2, \qquad G_{2,p}(u) = F_{2,p}(u) + \int_0^1 |u|^2,$$
$$G_{3,\varepsilon}(u) = F_{3,\varepsilon}(u) + \int_0^1 |u|^2,$$

their unique minimizers also converge in $L^2(0,1)$ to $\bar{v}$. This is an easy consequence of the fact that $C^\infty([0,1])$ is dense in $W^{1,1}(0,1)$.

It turns out that the minimizer $\bar{v}$ of (5.5) has a remarkable property:

Theorem 5.1. *The minimizer $\bar{v}$ of* (5.5) *is a constant K_i on each interval (a_i, a_{i+1}), $i = 0, 1, \cdots, k$ with the convention that $a_0 = 0$ and $a_{k+1} = 1$.*

Moreover

$$|K_i| \le 1/|a_{i+1} - a_i|, \qquad \forall i = 0, 1, \cdots, k. \tag{5.12}$$

The main ingredient in the proof of Theorem 5.1 is the following result taken from [3, Theorem 3] with roots in [6, Theorem 3.16].

Lemma 5.1. *Fix $\alpha, \beta, L \in \mathbb{R}$ and consider the minimization problem*

$$X = \operatorname{Inf}\left\{ \int_0^L |u'| + \int_0^L |u|^2; u \in BV(0,L),\ u(0) = \alpha \text{ and } u(L) = \beta \right\}. \tag{5.13}$$

A minimizer exists if and only if

$$\alpha = \beta \quad \text{with} \quad |\alpha| = |\beta| \le 1/L, \tag{5.14}$$

and in this case the unique minimizer in (5.13) *is the constant function $\alpha = \beta$.*

Proof. For the convenience of the reader we review briefly the argument from [3]. Set

$$H(u) = \int_0^L |u'| + \int_0^L |u|^2, \qquad u \in W^{1,1}(0,L), \tag{5.15a}$$

$$u(0) = \alpha, \qquad u(L) = \beta, \tag{5.15b}$$

and for $v \in BV(0,L)$,

$$H_r(v) = \text{ Inf} \liminf_{n\to\infty} H(v_n), \tag{5.16}$$

where the Inf in (5.16) is taken over all sequences $(v_n) \subset W^{1,1}(0,L)$ such that $v_n \to v$ in $L^2(0,L), v_n(0) = \alpha$ and $v_n(L) = \beta$.

From Lemmas 3.1 and 3.2 we know that

$$H_r(v) = \int_0^L |v'| + \int_0^L |v|^2 + |v(0) - \alpha| + |v(L) - \beta|, \qquad \forall v \in BV(0,L). \tag{5.17}$$

Moreover,

$$X = \min_{v \in BV} H_r(v). \tag{5.18}$$

Problem (5.13) usually admits *no* minimizer, while Problem (5.18) *always* admits a unique minimizer denoted $V \in BV(0,L)$. If (by chance!) Problem (5.13) admits a minimizer $U \in BV(0,L)$, then $U = V$. On the other hand, if we happen to know that the minimizer V of (5.18) satisfies $V(0) = \alpha$ and $V(L) = \beta$ then V is a minimizer for (5.13).

To summarize, the existence of a minimizer for (5.13) boils down to the question whether V satisfies $V(0) = \alpha$ and $V(L) = \beta$. We are thus led to study the properties of V. It is convenient to distinguish two cases:

Case 1: $\alpha\beta \le 0$. **Case 2**: $\alpha\beta > 0$.

In Case 1 we have $V \equiv 0$ and we conclude that our original Problem (5.13) admits a solution only if $\alpha = \beta = 0$; in this case $U \equiv 0$ is the minimizer of (5.13).

In Case 2 we may assume, without loss of generality, that

$$0 < \alpha \le \beta.$$

The heart of the matter is the surprising fact that V is a constant function (see the proof of Lemma 5.1 in [3]). In order to identify the constant we compute H_r given by (5.17) on the constant function $v \equiv t$; this yields

$$H_r(t) = Lt^2 + |t - \alpha| + |t - \beta|.$$

An easy inspection shows that $\min_{t \ge 0} H_r(t)$ is achieved at $t = 1/L$ if $\alpha > 1/L$ and at $t = \alpha$ if $\alpha \le 1/L$. □

Proof of Theorem 5.1. We apply Lemma 5.1 on each interval (a_i, a_{i+1}) with $L = a_{i+1} - a_i$, $\alpha = \bar{v}(a_i + 0)$ and $\beta = \bar{v}(a_{i+1} - 0)$. Clearly $\bar{v}$ restricted to (a_i, a_{i+1}) (and shifted) is a minimizer for (5.13). Otherwise we could find a function $w \in BV(a_i, a_{i+1})$ such that

$$\begin{aligned} &\int_{a_i}^{a_{i+1}} |w'| + \int_{a_i}^{a_i+1} |w|^2 < \int_{a_i}^{a_{i+1}} |\bar{v}'| + \int_{a_i}^{a_{i+1}} |\bar{v}|^2, \\ &w(a_i + 0) = \bar{v}(a_i + 0), \qquad w(a_{i+1} - 0) = \bar{v}(a_{i+1} - 0). \end{aligned}$$

Then the function $\bar{w} \in BV(0,1)$ defined by

$$\bar{w} := \begin{cases} w & \text{on } (a_i, a_{i+1}), \\ \bar{v} & \text{on } (0,1) \setminus (a_i, a_{i+1}), \end{cases}$$

would satisfy $G_r(\bar{w}) < G_r(\bar{v})$, which is imposible since $\bar{v}$ is a minimizer for G_r on $BV(0,1)$. We deduce from Lemma 5.1 that $\bar{v} = K_i$ on (a_i, a_{i+1}), for some constant K_i satisfying (5.12). □

As an immediate consequence of Theorem 5.1 we have now an explicit *finite-dimensional* convex minimization problem which governs Problem (5.5):

Corollary 5.1. *The unique minimizer $\bar{v}$ of* (5.5) *is given by*

$$\bar{v} = \sum_{i=0}^{k} \bar{K}_i \mathbb{1}_{(a_i, a_{i+1})} \tag{5.19}$$

and the constants $\bar{K}_i$ are obtained by minimizing

$$\Psi(K) = \sum_{i=0}^{k-1} |K_{i+1} - K_i| + \sum_{i=1}^{k} \varphi(dist\ (f_i, J_i) + \sum_{i=0}^{k} K_i^2 (a_{i+1} - a_i) \tag{5.20}$$

over $K = (K_0, \cdots, K_k) \in \mathbb{R}^{k+1}$, where J_i denotes the interval $[\min(K_{i-1}, K_i), \max(K_{i-1}, K_i)]$.

Finally we return to Problem (5.1) and derive some *necessary conditions* for the existence of a minimizer.

Corollary 5.2. *Assume that Problem* (5.1) *admits a minimizer $\bar{u} \in W^{1,1}(0,1)$, then necessarily*

$$\bar{u} \equiv \bar{K} \equiv \frac{1}{k+1} \sum_{i=1}^{k} f_i. \tag{5.21}$$

Moreover we must have

$$|f_i - \bar{K}| \le 1, \qquad \forall i = 1, \cdots, k, \tag{5.22a}$$

$$(a_{i+1} - a_i)|\bar{K}| \le 1, \qquad \forall i = 0, 1, \cdots, k, \tag{5.22b}$$

so that in particular

$$|\bar{K}| \le k + 1. \tag{5.23}$$

Proof of Corollary 5.2. Since $\bar{u}$ is also a minimizer for (5.5) we know by Theorem 5.1 that $\bar{u}$ is constant on each interval (a_i, a_{i+1}). On the other hand $\bar{u} \in W^{1,1}(0,1)$, and thus

$$\bar{u} \equiv \bar{K} \quad \text{on} \quad (0,1),$$

for some constant $\bar{K}$. To identify $\bar{K}$ we write that

$$G(\bar{K}) \leq G(t), \qquad \forall t \in \mathbb{R},$$

i.e.,

$$\sum_{i=1}^{k} |\bar{K} - f_i|^2 + |\bar{K}|^2 \leq \sum_{i=1}^{k} |t - f_i|^2 + t^2, \qquad \forall t \in \mathbb{R},$$

which implies (5.21). Next we recall that, by (5.7),

$$G(\bar{u}) = G_r(\bar{u}).$$

Going back to (5.1) and (5.4) we see that

$$F(\bar{u}) = F_r(\bar{u}),$$

which implies (5.22a) by (4.2). Finally (5.22b) comes from (5.12). □

Remark 5.2. In view of Corollary 5.2 it is easy to construct examples where Problem (5.2) admits no minimizer. Take for example $k = 2$ and f_1, f_2 such that $|2f_1 - f_2| > 3$.

6 Further directions of research

6.1) Try to adapt results from the previous sections to the following situations:
6.1.a) Let μ be a probability measure on $[0,1]$ and let

$$F(u) = \int_0^1 |u'| + \int_0^1 |u - f|^2 d\mu, \qquad u \in W^{1,1}(0,1),$$

where f is a given (smooth) function on $[0,1]$.
6.1.b) Let

$$F(u) = \int_0^1 (1 + |u'|^2)^{1/2} + \sum_{i=1}^{k} |u(a_i) - f_i|^2, \qquad u \in W^{1,1}(0,1),$$

where (a_i) and (f_i) are as in Section 1.
6.2) Investigate the following minimization problem:

$$\text{Inf} \left\{ \sum_{i=1}^{k} |u(a_i) - f_i|^2;\ u \in W^{1,1}(0,1),\ \int_0^1 |u'| \leq A,\ \left(\text{resp. } \int_0^1 (|u'| + |u|^2) \leq A\right) \right\},$$

where $A > 0$ is given.
6.3) Let Γ be a smooth curve in a domain $\Omega \subset \mathbb{R}^2$ and let

$$F(u) = \int_\Omega |\nabla u| + \int_\Gamma |u - f| d\sigma, \qquad u \in W^{1,1}(\Omega),$$

where f is a given (smooth) function on Γ. Study the minimization of F.

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