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Stability of T. Chan's Preconditioner from Numerical Range[†]

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Abstract. A matrix is said to be stable if the real parts of all the eigenvalues are negative. In this paper, for any matrix A_n, we discuss the stability properties of T. Chan's preconditioner $c_U(A_n)$ from the viewpoint of the numerical range. An application in numerical ODEs is also given.

Key words: T. Chan's preconditioner; stability; numerical range; boundary value method.

AMS subject classifications: 65F10, 65F15, 65L05, 65FN22

1 Introduction

T. Chan [9] proposed a circulant preconditioner for Toeplitz matrices in 1988. R. Chan, Jin and Yeung [6] showed that T. Chan's preconditioner can be defined not only for Toeplitz matrices but also for general matrices. Given a unitary matrix $U \in \mathbb{C}^{n\times n}$, define

$$\mathcal{M}_U \equiv \{U^*\Lambda_n U \mid \Lambda_n \text{ is any } n\text{-by-}n \text{ diagonal matrix}\}. \tag{1}$$

For any matrix $A_n \in \mathbb{C}^{n\times n}$, T. Chan's preconditioner $c_U(A_n) \in \mathcal{M}_U$ is defined to the minimizer of

$$\|c_U(A_n) - A_n\| = \min_{W_n \in \mathcal{M}_U} \|W_n - A_n\|,$$

where $\|\cdot\|$ is the Frobenius norm. Let F denote the Fourier matrix whose entries are given by:

$$(F)_{j,k} = \frac{1}{\sqrt{n}} e^{2\pi \mathbf{i}(j-1)(k-1)/n}, \qquad \mathbf{i} \equiv \sqrt{-1}, \quad 1 \le j,k \le n. \tag{2}$$

When $U = F$ in (1), $\mathcal{M}_U$ is the set of all circulant matrices [11]. It is proved that T. Chan's circulant preconditioner is a good preconditioner for solving a large class of linear systems, see, e.g., [4,5,7,9,16,17].

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In this paper, we will study some stability properties of T. Chan's preconditioner from the viewpoint of numerical range. The stability property is essential in many applications, including control theory and dynamical systems [1]. We first introduce the following definition.

Definition 1.1. A matrix is said to be stable if the real parts of all the eigenvalues are negative.

We now briefly review some important results. For any matrix $E \in \mathbb{C}^{n\times n}$, let $\delta(E)$ denote a diagonal matrix whose diagonal is equal to the diagonal of E. For T. Chan's preconditioner, we have the following lemma, see [6, 15, 22].

Lemma 1.1. *Let $A_n \in \mathbb{C}^{n\times n}$ and $c_U(A_n)$ be T. Chan's preconditioner. Then*

(i) $c_U(A_n)$ is uniquely determined by A_n and is given by

$$c_U(A_n) \equiv U^*\delta(UA_nU^*)U.$$

(ii) If A_n is Hermitian, then $c_U(A_n)$ is also Hermitian. Moreover, we have

$$\min_j \lambda_j(A_n) \le \min_j \lambda_j\big(c_U(A_n)\big) \le \max_j \lambda_j\big(c_U(A_n)\big) \le \max_j \lambda_j(A_n),$$

where $\lambda_j(E)$ is the j-th eigenvalue of E.

From Lemma 1.1 (ii), it is easy to see that if A_n is Hermitain and stable, then so is $c_U(A_n)$. In [19], Jin et al. showed that if A_n is normal and stable, then $c_U(A_n)$ is also normal and stable. The result is further generalized in [3]. It is proved that if A_n is $*$-congruent to a stable diagonal matrix, i.e., $A_n = Q^*DQ$ where Q is a nonsingular matrix and D is a stable diagonal matrix, then $c_U(A_n)$ is stable. Recently, by noting that any matrix A_n can be written as

$$A_n = H + \mathbf{i}K,$$

where

$$H = \frac{1}{2}(A_n + A_n^*) \quad \text{and} \quad K = \frac{1}{2\mathbf{i}}(A_n - A_n^*)$$

are Hermitian, Cheng and Jin proved the following result:

Lemma 1.2. ([10]) *Let $A_n \in \mathbb{C}^{n\times n}$ and suppose that $A_n = H + \mathbf{i}K$ where H and K are Hermitian. Then T. Chan's preconditioner $c_U(A_n)$ is stable for any unitary matrix $U \in \mathbb{C}^{n\times n}$ if and only if H is negative definite.*

It is a well-known fact that $*$-congruence does not change the inertia of a Hermitian matrix. Furthermore, for Hermitian matrices H and K with H nonsingular, H and K are simultaneously diagonalizable by $*$-congruence if and only if $H^{-1}K$ has real eigenvalues and is diagonalizable [13, p.229]. Suppose now that H is positive definite. Then $H^{-1}K$ is similar to $H^{-1/2}KH^{-1/2}$ which is Hermitian. Therefore, H and K are simultaneously diagonalizable by $*$-congruence. Of course, the same conclusion holds when H is negative definite. Thus, by using Lemma 1.2, one can show that the condition in [3], i.e., A_n is $*$-congruent to a stable diagonal matrix, is actually a necessary and sufficient condition for $c_U(A_n)$ to be stable for all unitary U. However, the condition in Lemma 1.2 is much simpler.

Another result concerning the stability of $c_U(A)$ is the following:

Lemma 1.3. ([10]) *Let $A_n \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $c_U(A_n)$ is stable if and only if*

$$\mathrm{Re}[\mathrm{tr}(A_n)] < 0,$$

where $\mathrm{Re}[\cdot]$ *denotes the real part of a complex number and* $\mathrm{tr}(\cdot)$ *denotes the trace of a matrix.*

Note that the solution of $A_n x = b$ is just the solution of

$$\theta A_n x = \theta b,$$

where θ is any nonzero complex number. Thus, for solving the system, we are free to consider θA_n instead of A_n. When $\mathrm{tr}(A_n)$ is nonzero, we can always find $\theta \in \mathbb{C}^n$ with $|\theta| = 1$ such that

$$\mathrm{Re}[\mathrm{tr}(\theta A_n)] = \mathrm{Re}[\theta \mathrm{tr}(A_n)] < 0.$$

Thus by Lemma 1.3, we have

Corollary 1.1. *Let $A_n \in \mathbb{C}^{n\times n}$. Then there exists θ with $|\theta| = 1$ and a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that T. Chan's preconditioner $c_U(\theta A_n)$ is stable if and only if $\mathrm{tr}(A_n) \neq 0$.*

T. Chan's preconditioner $c_U(A_n)$ is always normal for any unitary matrix U. In some cases, like the circulant preconditioner $c_F(A_n)$ where F is the Fourier matrix given by (2), we have definite formulas for their eigenvalues [11]. If all the eigenvalues are contained in some open half-plane defined by a line through the origin, we can always find θ with $|\theta| = 1$ such that all the eigenvalues of $\theta c_F(A_n)$ have negative real parts. Again, as far as stability of the preconditioner is concerned, one should replace the original system $A_n x = b$ by $\theta A_n x = \theta b$ because $c_F(\theta A_n) = \theta c_F(A_n)$ is stable.

2 Stability from numerical range

One may ask if we can have a result similar to Lemma 1.2 by using θA_n. To this end, we introduce the numerical range of A_n denoted by $W(A_n)$.

Definition 2.1. ([12, 14]) The numerical range of a matrix $A_n \in \mathbb{C}^{n\times n}$ is given by

$$W(A_n) \equiv \{x^* A_n x : x \in \mathbb{C}^n \text{ and } \|x\| = 1\} \subset \mathbb{C}.$$

The following lemma is essential in the study of numerical range, see [12, 14].

Lemma 2.1. *Let $A_n \in \mathbb{C}^{n\times n}$. We have*

(i) $W(U^* A_n U) = W(A_n)$ for any unitary matrix $U \in \mathbb{C}^{n\times n}$.

(ii) $W(A_n)$ is a compact convex set.

(iii) $\sigma(A_n) \subset W(A_n)$ where $\sigma(A_n)$ is the spectrum of the matrix A_n.

(iv) If A_n is diagonal, then $W(A_n)$ is the convex hull of the diagonal entries of A_n.

For a convex set S in $\mathbb{C}$, it is easy to obtain the following lemma.

Lemma 2.2. *For a convex set S in $\mathbb{C}$, there exists an θ with $|\theta| = 1$ such that $\theta S \subset \mathbb{C}^-$ if and only if $0 \notin S$, where $\mathbb{C}^- \equiv \{q \in \mathbb{C} : \mathrm{Re}(q) < 0\}$.*

Proof If $0 \notin S$, by the convexity of S, we know that there is a straight line l passing through the origin and that S is contained in one of the two open half-planes determined by l. Consequently, there exists a complex number θ with $|\theta| = 1$ such that $\theta S \subset \mathbb{C}^-$. It is also true conversely. ■

We therefore have

Theorem 2.1. *Let $A_n \in \mathbb{C}^{n\times n}$. Then there exists some θ with $|\theta| = 1$ such that T. Chan's preconditioner $c_U(\theta A_n)$ is stable for all unitary matrices $U \in \mathbb{C}^{n\times n}$ if and only if $0 \notin W(A_n)$.*

Proof " $\Rightarrow$ ": Suppose that there exists some θ with $|\theta| = 1$ such that $c_U(\theta A_n)$ is stable for all unitary matrices $U \in \mathbb{C}^{n\times n}$. By Lemma 1.2, we know that if we write

$$\theta A_n = H + \mathbf{i}K,$$

where H and K are Hermitian, then H is negative definite and so $x^*Hx < 0$ for all nonzero $x \in \mathbb{C}^n$. Thus,

$$0 \notin \{x^*Hx + \mathbf{i}x^*Kx : x \in \mathbb{C}^n \text{ and } \|x\| = 1\} = W(\theta A_n).$$

Note that

$$W(\theta A_n) = \theta W(A_n)$$

and thus $0 \notin W(A_n)$.

" $\Leftarrow$ ": Suppose $0 \notin W(A_n)$. Then by Lemma 2.1 (ii) and Lemma 2.2, there is a complex number θ with $|\theta| = 1$ such that

$$W(\theta A_n) = \theta W(A_n) \subset \mathbb{C}^-.$$

Let θA_n be decomposed as

$$\theta A_n = H + \mathbf{i}K,$$

where H and K are Hermitian. We have

$$\{x^*Hx + \mathbf{i}x^*Kx : x \in \mathbb{C}^n \text{ and } \|x\| = 1\} = W(\theta A_n) \subset \mathbb{C}^-$$

which implies that $x^*Hx < 0$ for all $x \in \mathbb{C}^n$ with $\|x\| = 1$ and so H is negative definite. By Lemma 1.2, $c_U(\theta A_n)$ is stable for all unitary matrices $U \in \mathbb{C}^{n\times n}$. ■

Now, the problem we are facing is that how to judge $0 \notin W(A)$. We remark that when $A \in \mathbb{C}^{2\times 2}$, $W(A)$ is always an elliptical disk, possibly degenerate. Moreover, one can always find the (boundary) ellipse in terms of the 4 entries of A. For a general matrix $A \in \mathbb{C}^{n\times n}$ and $x, y \in \mathbb{C}^n$, let

$$A_{xy} = \begin{bmatrix} x^*Ax & x^*Ay \\ y^*Ax & y^*Ay \end{bmatrix}_{2\times 2}.$$

It is proved in [20] that $W(A)$ is the union of all the sets $W(A_{xy})$ where x and y run over all pairs of real orthonormal vectors. Thus we can have a rough picture about the shape of $W(A)$ by plotting enough $W(A_{xy})$.

When A_n is Hermitian, $W(A_n)$ is the interval with endpoints being the largest and smallest eigenvalues of A_n. Thus, Lemma 1.1 (ii) means that when A_n is Hermitian, $W(c_U(A_n)) \subset W(A_n)$. In fact, the result is true for any matrix A_n.

Theorem 2.2. *For any matrix $A_n \in \mathbb{C}^{n\times n}$ and any unitary matrix $U \in \mathbb{C}^{n\times n}$, we have*

$$W(c_U(A_n)) \subset W(A_n).$$

Proof By Lemma 1.1 (i) and Lemma 2.1 (i), we have

$$W(c_U(A_n)) = W(U^*\delta(UA_nU^*)U) = W(\delta(UA_nU^*)).$$

Let $\delta(UA_nU^*)_{ii} = d_{ii}$, for $i = 1, 2, \cdots, n$. By direct calculation and Lemma 2.1 (i), we have

$$d_{ii} \in W(UA_nU^*) = W(A_n). \tag{3}$$

Note that by Lemma 2.1 (iv),

$$W(\delta(UA_nU^*)) = \mathrm{Co}(\{d_{11}, d_{22}, \cdots, d_{nn}\}), \tag{4}$$

where $\mathrm{Co}(S)$ denoted the convex hull of S. Since $W(A_n)$ is convex, we have by (3) and (4),

$$W(\delta(UA_nU^*)) \subset W(A_n),$$

i.e.,

$$W(c_U(A_n)) \subset W(A_n). \quad \blacksquare$$

By Theorem 2.2 and Lemma 2.1 (iii), we get, for any unitary U,

$$\sigma(c_U(A_n)) \subset W(c_U(A_n)) \subset W(A_n).$$

Thus, if we can have $W(A_n) \subset \mathbb{C}^-$ (equivalently H negative definite), then $c_U(A_n)$ is stable for all unitary U. Due to the convexity of $W(A_n)$, if $0 \notin W(A_n)$, we know that there exists θ such that $W(\theta A_n) \subset \mathbb{C}^-$ and so $c_U(\theta A_n)$ is stable for all unitary U. For example,

(i) If H is positive definite, then $c_U(-A_n)$ is stable for all unitary U;

(ii) If K is positive definite, then $c_U(\mathbf{i}A_n)$ is stable for all unitary U;

(iii) If K is negative definite, then $c_U(-\mathbf{i}A_n)$ is stable for all unitary U.

More generally, with $\beta = \cos\alpha + \mathbf{i}\sin\alpha$,

$$\frac{1}{2}[(\beta A_n) + (\beta A_n)^*] = (\cos\alpha)H - (\sin\alpha)K.$$

Thus, we can deduce that if there exist $a, b \in [-1, 1]$ such that $aH + bK$ is positive definite, then there exists some θ with $|\theta| = 1$ such that $c_U(\theta A_n)$ is stable for all unitary U.

3 An application

In [8, 16, 18], the following initial value problem is considered

$$\begin{cases} \dfrac{d\mathbf{y}(t)}{dt} & = & \mathbf{f}(t) \equiv J\mathbf{y}(t) + \mathbf{g}(t), \quad t \in (t_0, T], \\ \mathbf{y}(t_0) & = & \mathbf{z}, \end{cases} \tag{5}$$

where $\mathbf{y}(t)$, $\mathbf{f}(t)$, $\mathbf{g}(t) : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^m$ and $J \in \mathbb{R}^{m\times m}$. Let the grid points be given by

$$t_j = t_0 + jh, \quad j = 0, \cdots, s,$$

where $h = (T - t_0)/s$. To get $y(t_j)$, $j = 1, 2, \cdots, s$, the boundary value method (BVM) is used.

3.1 Boundary value method

The BVM is based on k-step linear multistep formula:

$$\sum_{i=-\nu}^{k-\nu} \alpha_{i+\nu}\mathbf{y}_{n+i} = h\sum_{i=-\nu}^{k-\nu} \beta_{i+\nu}\mathbf{f}_{n+i}, \quad n = \nu, \ldots, s-k+\nu, \tag{6}$$

and boundary values:

$$\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_\nu, \quad \mathbf{y}_{s-k+\nu+1}, \mathbf{y}_{s-k+\nu+2}, \ldots, \mathbf{y}_s. \tag{7}$$

Note that in (7), only $\mathbf{y}_0$ is known. For the remainders in (7), we use other two sets of additional equations with the same order of accuracy of (6),

$$\sum_{i=0}^{k} \alpha_i^{(j)}\mathbf{y}_i = h\sum_{i=0}^{k} \beta_i^{(j)}\mathbf{f}_i, \qquad j = 1, \cdots, \nu-1, \tag{8}$$

and

$$\sum_{i=0}^{k} \alpha_{k-i}^{(j)}\mathbf{y}_{s-i} = h\sum_{i=0}^{k} \beta_{k-i}^{(j)}\mathbf{f}_{s-i}, \qquad j = s-k+\nu+1, \cdots, s, \tag{9}$$

see [2] for a detail. By combining (6), (8), (9), and the initial value $\mathbf{y}_0$, we obtain a linear system:

$$M\mathbf{y} = \mathbf{b}, \tag{10}$$

where

$$M = A \otimes I - hB \otimes J \tag{11}$$

with $I \in \mathbb{R}^{m\times m}$ being the identity matrix, J being the matrix from (5), and " $\otimes$ " being the Kronecker product. The vector $\mathbf{y}$ in (10) is defined by

$$\mathbf{y}^T = [\mathbf{y}_0^T, \mathbf{y}_1^T, \ldots, \mathbf{y}_s^T] \in \mathbb{R}^{m(s+1)}.$$

The known vector $\mathbf{b} \in \mathbb{R}^{m(s+1)}$ in (10) depends on $\mathbf{f}$, the boundary values and the coefficients of the method. The matrix $A \in \mathbb{R}^{(s+1)\times(s+1)}$ in (11) is defined as follows,

$$A = \begin{bmatrix} 1 & \cdots & 0 & & \\ {\alpha_0}^{(1)} & \cdots & {\alpha_k}^{(1)} & & \\ \vdots & \vdots & \vdots & & \\ {\alpha_0}^{(\nu-1)} & \cdots & {\alpha_k}^{(\nu-1)} & \mathbf{O} & \\ \alpha_0 & \cdots & \alpha_k & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_0 & \cdots & \alpha_k \\ & \mathbf{O} & {\alpha_0}^{(s-k+\nu+1)} & \cdots & {\alpha_k}^{(s-k+\nu+1)} \\ & & \vdots & \vdots & \vdots \\ & & \alpha_0^{(s)} & \cdots & {\alpha_k}^{(s)} \end{bmatrix}. \tag{12}$$

The matrix B in (11) is defined similarly by using $\beta' s$ instead of $\alpha' s$ in (12) and the first row of B is zero, see [2,4,8]. Let

$$A_T = \begin{bmatrix} \alpha_\nu & \cdots & \alpha_k & & \\ \vdots & \ddots & \ddots & \ddots & \\ \alpha_0 & \ddots & \ddots & \ddots & \alpha_k \\ & \ddots & \ddots & \ddots & \vdots \\ & & \alpha_0 & \cdots & \alpha_\nu \end{bmatrix} \in \mathbb{R}^{(s+1)\times(s+1)}$$

and $B_T \in \mathbb{R}^{(s+1)\times(s+1)}$ is defined similarly by using $\beta' s$ instead of $\alpha' s$ in A_T. Then, the following preconditioner is considered:

$$S \equiv s(A_T) \otimes I - hs(B_T) \otimes c_F(J), \tag{13}$$

where $s(A_T)$, $s(B_T)$ are Strang's circulant preconditioners for A_T, B_T respectively, and $c_F(J)$ is T. Chan's circulant preconditioner for J. We remark that for a Toeplitz matrix $T_n = [t_{ij}]$ with $t_{ij} = t_{i-j}$, the diagonals of Strang's circulant preconditioner $s(T_n)$ are defined by

$$s_k = \begin{cases} t_k, & 0 \le k \le \lfloor n/2 \rfloor, \\ t_{k-n}, & \lfloor n/2 \rfloor < k < n, \\ s_{n+k}, & -n < k < 0. \end{cases}$$

To show that S is invertible, we need to introduce the stability of the BVM. The characteristic polynomials $\rho(z)$ and $\sigma(z)$ of the BVM are defined by

$$\rho(z) \equiv z^\nu \sum_{j=\nu}^{k-\nu} \alpha_{j+\nu} z^j \quad \text{and} \quad \sigma(z) \equiv z^\nu \sum_{j=\nu}^{k-\nu} \beta_{j+\nu} z^j,$$

where $\{\alpha_i\}$, $\{\beta_i\}$ are given by (6). The $A_{\nu,k-\nu}$-stability polynomial is defined by

$$\pi(z,q) \equiv \rho(z) - q\sigma(z),$$

where z, $q \in \mathbb{C}$.

Definition 3.1. ([2]) The region

$$\mathcal{D}_{\nu,k-\nu} = \{q \in \mathbb{C} : \pi(z,q) \text{ has } \nu \text{ zeros inside } |z| = 1 \text{ and } k-\nu \text{ zeros outside } |z| = 1\}$$

is called the region of $A_{\nu,k-\nu}$-stability for a given BVM with $(\nu, k-\nu)$-boundary conditions. Moreover, the BVM is said to be $A_{\nu,k-\nu}$-stable if $\mathbb{C}^- \subseteq \mathcal{D}_{\nu,k-\nu}$.

We have the following theorem on the invertibility of the preconditioner S. The proof is similar to that of Theorem 2 in [3], and we therefore omit it. Nevertheless, we note here that the stability of $c_F(J)$ is crucial in the proof and it is now ensured by a much simpler condition, namely $J + J^T$ is negative definite.

Theorem 3.1. *If the BVM for* (5) *is $A_{\nu,k-\nu}$-stable and the matrix $J + J^T$ is negative definite, then the preconditioner S in* (13) *is invertible.*

Table 1: Number of iterations and CPU time (sec.).

		No. of iterations		CPU time	
m	s	I	S	I	S
24	20	80	4	1.469	0.140
	40	160	4	6.719	0.250
	80	320	4	53.063	0.469
48	20	80	4	2.890	0.250
	40	160	4	19.250	0.484
	80	320	4	113.500	0.969
96	20	80	4	6.422	0.500
	40	160	4	34.391	0.938
	80	320	4	273.891	1.969

3.2 Numerical test

To illustrate the efficiency of our proposed preconditioner S defined by (13), one numerical example is given in this section. The BVM we used here is the fifth order generalized Adams method [2]. All experiments are performed in MATLAB and the M-file "gmres" is used to solve the preconditioned systems. In our calculations, the stopping criterion in the GMRES method [21] is

$$\frac{\|\mathbf{r}_q\|_2}{\|\mathbf{r}_0\|_2} < 10^{-6},$$

where $\mathbf{r}_q$ is the residual after the q-th iteration and the zero vector is the initial guess. All programs are run on a 2.4 GHz PC with 1.024 Gbytes of memory.

Example 3.1. Consider

$$\begin{cases} \dfrac{d\mathbf{y}(t)}{dt} = Q_m \mathbf{y}(t), & t \in (0,1]\,, \\[2ex] \mathbf{y}(0) = (1,1,\cdots,1)^T, \end{cases}$$

where $Q_m = [q_{ij}]_{i,j=1}^m$ with

$$q_{ij} = \begin{cases} -3, & i = j; \\[1ex] 2^{-(i+j-1)}, & i > j; \\[1ex] 3^{-(i+j-1)}, & i < j. \end{cases}$$

It is easy to check that $Q_m + Q_m^T$ is negative definite. Therefore by Theorem 3.1, the preconditioner S defined in (13) is invertible. Table 1 shows the number of iterations and CPU time in seconds required for convergence with different combinations of matrix sizes m and s. In the table, I denotes no preconditioner and S is our new preconditioner. As expected, the number of iterations required for convergence is small and remains a constant for increasing m and s with the preconditioner S. The CPU time required for convergence with the preconditioner S is again much less than that without preconditioner, especially for large values of m and s.

4 Concluding remarks

In this paper, we have used the concept of numerical range to study some of the stability properties of T. Chan's preconditioner. T. Chan's preconditioner $c_U(A_n)$ is a normal matrix with eigenvalues being the diagonal elements of UA_nU^* and the numerical range $W(A_n)$ is exactly the set of all possible diagonal elements of UA_nU^* when U runs through all the unitary matrices. It is quite natural that these two subjects are closely related. The numerical range, together with its generalizations, is a rich subject (see [12] and [14, Chapter 1]) and has been studied intensively. Here we have used only the most basic properties to obtain some elementary results. It is expected that more results on $c_U(A)$ can be obtained.

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