# OPTIMAL POINT-WISE ERROR ESTIMATE OF A COMPACT FINITE DIFFERENCE SCHEME FOR THE COUPLED NONLINEAR SCHRÖDINGER EQUATIONS*

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### Abstract

In this paper, we analyze a compact finite difference scheme for computing a coupled nonlinear Schrödinger equation. The proposed scheme not only conserves the total mass and energy in the discrete level but also is decoupled and linearized in practical computation. Due to the difficulty caused by compact difference on the nonlinear term, it is very hard to obtain the optimal error estimate without any restriction on the grid ratio. In order to overcome the difficulty, we transform the compact difference scheme into a special and equivalent vector form, then use the energy method and some important lemmas to obtain the optimal convergent rate, without any restriction on the grid ratio, at the order of $O(h^4 + \tau^2)$ in the discrete $L^\infty$-norm with time step $\tau$ and mesh size $h$. Finally, numerical results are reported to test our theoretical results of the proposed scheme.

*Mathematics subject classification:* 65M06, 65M12.
*Key words:* Coupled nonlinear Schrödinger equations, Compact difference scheme, Conservation, Point-wise error estimate.

## 1. Introduction

In this paper, we consider the coupled nonlinear Schrödinger (CNLS) equations

$$i\partial_t u + k\partial_{xx} u + (|u|^2 + \beta|v|^2)u = 0, \quad x \in \Omega \subseteq \mathbb{R}, \ t > 0, \tag{1.1}$$

$$i\partial_t v + k\partial_{xx} v + (|v|^2 + \beta|u|^2)v = 0, \quad x \in \Omega \subseteq \mathbb{R}, \ t > 0, \tag{1.2}$$

which arise in a great variety of physical situations. In fiber communication system, such equations have been shown to govern pulse propagation along orthogonal polarization axes in nonlinear optical fibers and in wavelength-division-multiplexed systems [25,32,41]. Here $u(x,t)$ and $v(x,t)$ are unknown complex-valued wave functions, $k$ describes the dispersion in the optic fiber, $\beta$ is defined for birefringent optic fiber coupling parameter, $\Omega$ is a bounded computational domain, $i$ is the imaginary unit, i.e. $i^2 = -1$. These equations also model two-component Bose-Einstein condensation and beam propagation inside crystals, photorefractives as well as water wave interactions.

There are many studies on numerical studying of the CNLS equations. In [1, 2, 42], some efficient time-splitting spectral methods were given to study the dynamics of two-component Bose-Einstein condensate. In [30], a multi-symplectic method was constructed and the solitons collision was simulated. In [29], a nonlinear implicit conservative scheme was proposed for the

strong coupling of Schrödinger equations and both the analytic and numerical solutions were discussed. In [15–17] a Crank-Nicolson difference scheme, a linearized implicit scheme and a compact difference scheme were presented and some numerical experiments were given. In [13], Ismail discretized the space derivative by central difference formulas of fourth-order, then solved the resulting ordinary differential system by the fourth-order explicit Runge-Kutta method. The linearly convergence of all of the difference schemes in [13,15–17] was proved by *von Neumann* method. In [14], Galerkin finite element method was proposed to solve the CNLS equation. In [33], Wang discussed the splitting spectral method for solving the CNLS equation. In [35], Wang et al. proposed and studied a nonlinear symplectic difference scheme. They proved the existence, uniqueness and second order convergence in $l^2$-norm under some restrictions on the grid ratios, and proposed an iterative algorithm for solving the difference scheme. In [31], Sun and Zhao also studied the nonlinear difference scheme proposed in [16,28,29]. They proved the existence, uniqueness and second order convergence in $l^\infty$ norm (the discrete $L^\infty$ norm), and proposed another interesting iterative algorithm for solving the nonlinear scheme. In [36], the optimal error estimate in $l^\infty$ norm of the linearized difference scheme proposed in [17,34] was established.

Recently, there has been growing interest in high-order compact methods for solving partial differential equations, see, e.g., [4–7, 9–12, 18–23, 26, 40, 43]. It was shown that the high-order difference methods play an important role in the simulation of high frequency wave phenomena. However, due to the difficulty caused by the compact difference on the nonlinear term, the energy method can not be used directly on the compact difference scheme, and so there is few proof of the unconditional error estimate in the $l^\infty$-norm of any a compact difference scheme for nonlinear partial differential equations. In [37–39], without any restrictions on the grid ratios, we established the optimal $l^\infty$-error estimates of some compact difference schemes for the nonlinear Schrödinger equation (NLSE) with *periodic* boundary conditions where the circulant coefficient matrix was used, but the technique can not be extended directly to NLSE with Dirichlet boundary condition because the coefficient matrix is no longer circulant.

In this paper, we introduce an efficient compact difference scheme for the CNLS equation on a finite domain $\Omega = [L_1, L_2]$ with initial conditions

$$u(x,0) = \psi(x), \quad v(x,0) = \phi(x), \quad x \in [L_1, L_2], \tag{1.3}$$

and homogeneous Dirichlet boundary conditions

$$u(L_1,t) = u(L_2,t) = 0, \quad v(L_1,t) = v(L_2,t) = 0, \quad t > 0, \tag{1.4}$$

where $\psi(x)$ and $\phi(x)$ are prescribed smooth functions vanishing at points $x = L_1$ and $x = L_2$.

The problem (1.1)-(1.4) has two kinds of standard conserved quantities, i.e., the total masses

$$M_1(t) := \int_{L_1}^{L_2} |u(x,t)|^2 dx \equiv M_1(0), \quad t \geq 0, \tag{1.5}$$

$$M_2(t) := \int_{L_1}^{L_2} |v(x,t)|^2 dx \equiv M_2(0), \quad t \geq 0, \tag{1.6}$$

and energy

$$E(t) := \frac{k}{2} \int_{L_1}^{L_2} \left( |\partial_x u(x,t)|^2 + |\partial_x v(x,t)|^2 \right) dx - \frac{1}{4} \int_{L_1}^{L_2} \left( |u(x,t)|^4 + |v(x,t)|^4 \right) dx$$

$$- \frac{\beta}{2} \int_{L_1}^{L_2} |u(x,t)|^2 |v(x,t)|^2 dx \equiv E(0), \quad t \geq 0. \tag{1.7}$$

Z. Fei. et al. pointed out in [8] that the nonconservative schemes may easily show non-linear blow-up, and they presented a new conservative linear difference scheme for nonlinear Schrödinger equation. In [24], Li and Vu-Quoc also said, "...in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation." However, the conservative difference schemes in existed references almost are of second-order accuracy. To construct and analyze a stable finite difference scheme which not only has high-order accuracy but also conserves the total masses and energy in the discrete level is an important and interesting topic.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and important lemmas, then propose a compact finite difference scheme for the problem (1.1)-(1.4) and state our main error estimate result. In Section 3, we prove that the proposed scheme conserves the total masses and energy in the discrete level, and establish the optimal $l^\infty$-norm error bound of the difference solutions by using *a priori* $l^\infty$-norm estimate of the numerical solution. In Section 4, numerical results are reported to support our error estimate. Finally, some concise conclusions are drawn in Section 5.

## 2. Finite Difference Scheme and Main Results

In this section, we introduce some notions and important lemmas, propose a compact finite difference scheme for the problem (1.1)-(1.4), and state our main error estimate result.

Before giving the finite difference scheme, some notations are introduced. For a positive integer $N$, choose time-step $\tau = T/N$ and denote time levels $t_n = n\tau$, $n = 0, 1, \cdots, N$, where $0 < T < T_{\max}$ with $T_{\max}$ the maximal existing time of the solution; choose mesh size $h = (L_2 - L_1)/J$, with $J$ a positive integer and denote grid points as $x_j = L_1 + jh$, $j = 0, 1, \cdots, J$.

Denote $(U_j^n, V_j^n)$ and $(u_j^n, v_j^n)$ be the numerical approximation, and respectively the exact solution of $(u(x_j, t_n), v(x_j, t_n))$ for $j = 0, 1, \cdots, J$ and $n = 0, 1, \cdots, N$, and denote $(U^n, V^n) \in \mathbb{C}^{J+1} \times \mathbb{C}^{J+1}$ be the numerical solution and respectively $(u^n, v^n) \in \mathbb{C}^{J+1} \times \mathbb{C}^{J+1}$ be the exact solution at time $t = t_n$. For a grid function $u = \{u_j^n \mid j = 0, 1, \cdots, J; \ n = 0, 1, \cdots, N\}$, introduce the following finite difference operators:

$$\delta_x^+ u_j^n = \frac{1}{h}\left(u_{j+1}^n - u_j^n\right), \ \ \delta_x^2 u_j^n = \frac{1}{h^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right), \ \ \delta_t^+ u_j^n = \frac{1}{\tau}\left(u_j^{n+1} - u_j^n\right),$$

$$\delta_t u_j^n = \frac{1}{2\tau}\left(u_j^{n+1} - u_j^{n-1}\right), \quad \mathcal{A}_h u_j^n = \frac{1}{12}\left(u_{j-1}^n + 10u_j^n + u_{j+1}^n\right).$$

We denote the space

$$X_h := \left\{u = (u_0, u_1, u_2, \cdots, u_J) \mid u_0 = u_J = 0\right\} \subseteq \mathbb{C}^{J+1},$$

$$X_h^0 := \left\{\hat{u} \mid \hat{u} = u(2:J) = (u_1, u_2, \cdots, u_{J-1}), \ u \in X_h\right\} \subseteq \mathbb{C}^{J-1},$$

and matrices

$$H = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 10 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 10 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 10 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 10 \end{pmatrix}_{(J-1)\times(J-1)},$$

$$B = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}_{(J-1)\times(J-1)} ,$$

and define discrete inner product and discrete norms over $X_h$ and $X_h^0$ as

$$\langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle_0 := h \sum_{j=1}^{J-1} u_j \overline{v}_j, \quad ||u|| := \langle u, v \rangle^{\frac{1}{2}}, \ ||u||_4 := \left( h \sum_{j=1}^{J-1} |u_j|^4 \right)^{\frac{1}{4}},$$

$$||u||_\infty := \max_{0 \le j \le J} |u_j|, \quad ||\delta_x^+ u|| := \left( h \sum_{j=0}^{J-1} |\delta_x^+ u_j|^2 \right)^{\frac{1}{2}}, \quad |||\delta_x^+ u||| := \langle AB\hat{u}, \hat{u} \rangle_0^{\frac{1}{2}},$$

where $\overline{f}$ denotes the conjugate of $f$, $A = H^{-1}$. It should be pointed out that $||\delta_x^+ u||$ and $|||\delta_x^+ u|||$ are semi-norms of the discrete function $u$. Throughout the paper, we denote $C$ as generic positive constant which may be dependent on the regularity of exact solution and the given data but independent of the time step $\tau$ and the grid size $h$, and we use the notation $w \lesssim v$ to present $w \le Cv$.

**Lemma 2.1.** *On the matrices $H$ and $B$, we have the following results:*
(a) *The eigenvalues of the matrices $H$ and $B$ are*

$$\lambda_{H,j} = \frac{1}{12} \left( 10 + 2\cos\frac{j\pi}{J} \right), \quad \lambda_{B,j} = \frac{1}{h^2} \left( 2 + 2\cos\frac{j\pi}{J} \right), \quad j = 1, \cdots, J-1. \tag{2.1}$$

(b) *They have the same eigenvectors, i.e.*

$$v_k = \left( \sin\frac{k\pi}{J}, \sin\frac{2k\pi}{J}, \cdots, \sin\frac{(J-1)k\pi}{J} \right)^\top, \quad k = 1, \cdots, J-1. \tag{2.2}$$

(c) $AB = BA$, $AD = DA$, *where* $A = H^{-1}$ *and* $D^2 = B$.

*Proof.* The results (a) and (b) can be verified directly. It follows from (b) that $HB = BH$ which implies $AB = BA$. Noticing that $B$ is symmetric and positive definite, there exits a symmetric and positive definite matrix $D$ such that $B = D^2$. Because $D$ and $D^2$ have same eigenvectors, the symmetric matrixes $D$ and $H$ have the same eigenvectors. This gives $DH = HD$ which implies $DA = AD$. $\qquad \square$

**Lemma 2.2.** *For any grid function $u \in X_h$, we have*

$$||\delta_x^+ u|| \le |||\delta_x^+ u||| \le \frac{\sqrt{6}}{2} ||\delta_x^+ u||. \tag{2.3}$$

*Proof.* Noticing $A = H^{-1}$, we obtain that the biggest eigenvalue of the matrix $A$ is $\lambda_{H,J-1}^{-1}$, and the smallest one is $\lambda_{H,1}^{-1}$. This, together with the definition of the discrete inner product

and semi-norms, gives

$$|||\delta_x^+ \hat{u}|||^2 = \langle AB\hat{u}, \hat{u}\rangle_0 = \langle AD\hat{u}, D\hat{u}\rangle_0 \leq \lambda_{H,J-1}^{-1}\langle D\hat{u}, D\hat{u}\rangle_0 \leq \frac{3}{2}\langle D\hat{u}, D\hat{u}\rangle_0$$

$$= \frac{3}{2}\langle B\hat{u}, \hat{u}\rangle_0 = -\frac{3}{2}\langle \delta_x^2 u, u\rangle = \frac{3}{2}h\sum_{j=0}^{J-1}|\delta_x^+ u_j|^2 = \frac{3}{2}||\delta_x^+ u||^2, \tag{2.4}$$

$$|||\delta_x^+ \hat{u}|||^2 = \langle AB\hat{u}, \hat{u}\rangle_0 = \langle AD\hat{u}, D\hat{u}\rangle_0 \geq \lambda_{H,1}^{-1}\langle D\hat{u}, D\hat{u}\rangle_0 \geq \langle D\hat{u}, D\hat{u}\rangle_0$$

$$= \langle B\hat{u}, \hat{u}\rangle_0 = -\langle \delta_x^2 u, u\rangle = h\sum_{j=0}^{J-1}|\delta_x^+ u_j|^2 = ||\delta_x^+ u||^2, \tag{2.5}$$

where Lemma 2.1 and the summation by parts formula were used. It follows from (2.4) and (2.5) that

$$||\delta_x^+ u|| \leq |||\delta_x^+ u||| \leq \frac{\sqrt{6}}{2}||\delta_x^+ u||.$$

This completes the proof.                                                                              $\square$

**Lemma 2.3.** *For any grid function* $u^n \in X_h$, $n = 0, 1, \cdots, N$, *we have*

$$\mathrm{Re}\langle AB(\hat{u}^{n+1} + \hat{u}^n), \hat{u}^{n+1} - \hat{u}^n\rangle_0 = |||\delta_x^+ u^{n+1}|||^2 - |||\delta_x^+ u^n|||^2. \tag{2.6}$$

*Proof.* By virtue of Lemma 2.1, we have

$$\mathrm{Re}\langle AB\hat{u}^{n+1}, \hat{u}^n\rangle_0 = \mathrm{Re}\langle \hat{u}^{n+1}, BA\hat{u}^n\rangle_0 = \mathrm{Re}\langle \hat{u}^{n+1}, AB\hat{u}^n\rangle_0 = \mathrm{Re}\langle AB\hat{u}^n, \hat{u}^{n+1}\rangle_0.$$

This gives

$$\mathrm{Re}\langle AB(\hat{u}^{n+1} + \hat{u}^n), \hat{u}^{n+1} - \hat{u}^n\rangle_0$$
$$= \mathrm{Re}\langle AB\hat{u}^{n+1}, \hat{u}^{n+1}\rangle_0 - \mathrm{Re}\langle AB\hat{u}^{n+1}, \hat{u}^n\rangle_0 + \mathrm{Re}\langle AB\hat{u}^n, \hat{u}^{n+1}\rangle_0 - \mathrm{Re}\langle AB\hat{u}^n, \hat{u}^n\rangle_0$$
$$= \mathrm{Re}\langle AB\hat{u}^{n+1}, \hat{u}^{n+1}\rangle_0 - \mathrm{Re}\langle AB\hat{u}^n, \hat{u}^n\rangle_0 = |||\delta_x^+ u^{n+1}|||^2 - |||\delta_x^+ u^n|||^2.$$

This completes the proof.                                                                              $\square$

**Lemma 2.4.** ([27]) *For any grid function* $u \in X_h$, *there are*

$$||u|| \lesssim ||\delta_x^+ u||, \quad ||u||_\infty^2 \lesssim ||u|| \, ||\delta_x^+ u||, \quad ||u||_\infty \lesssim ||\delta_x^+ u||. \tag{2.7}$$

For computing the problem (1.1)-(1.4), we consider the following finite difference scheme:

$$i\mathcal{A}_h\delta_t U_j^n + \frac{k}{2}\delta_x^2\left(U_j^{n-1} + U_j^{n+1}\right) + \frac{1}{2}\mathcal{A}_h\left(\left(|U_j^n|^2 + \beta|V_j^n|^2\right)\left(U_j^{n-1} + U_j^{n+1}\right)\right) = 0,$$
$$j = 1, \cdots, J-1; \ n = 1, \cdots, N-1, \tag{2.8}$$

$$i\mathcal{A}_h\delta_t V_j^n + \frac{k}{2}\delta_x^2\left(V_j^{n-1} + V_j^{n+1}\right) + \frac{1}{2}\mathcal{A}_h\left(\left(|V_j^n|^2 + \beta|U_j^n|^2\right)\left(V_j^{n-1} + V_j^{n+1}\right)\right) = 0,$$
$$j = 1, \cdots, J-1; \ n = 1, \cdots, N-1, \tag{2.9}$$

$$i\mathcal{A}_h\delta_t^+ U_j^0 + \frac{k}{2}\delta_x^2\left(U_j^0 + U_j^1\right) + \frac{1}{4}\mathcal{A}_h\Big(\left(|U_j^0|^2 + |U_j^1|^2 + \beta|V_j^0|^2 + \beta|V_j^1|^2\right)$$
$$\times \left(U_j^0 + U_j^1\right)\Big) = 0, \quad j = 1, \cdots, J - 1, \tag{2.10}$$

$$i\mathcal{A}_h\delta_t^+ V_j^0 + \frac{k}{2}\delta_x^2\left(V_j^0 + V_j^1\right) + \frac{1}{4}\mathcal{A}_h\Big(\left(|V_j^0|^2 + |V_j^1|^2 + \beta|U_j^0|^2 + \beta|U_j^1|^2\right)$$
$$\times \left(V_j^0 + V_j^1\right)\Big) = 0, \quad j = 1, \cdots, J - 1, \tag{2.11}$$

$$U_j^0 = \psi(x_j), \quad V_j^0 = \phi(x_j), \quad j = 0, 1, \cdots, J, \tag{2.12}$$

$$U^n \in X_h, \quad V^n \in X_h, \quad n = 1, \cdots, N. \tag{2.13}$$

The vector form of the scheme is

$$iH\delta_t\hat{U}^n - \frac{k}{2}B\left(\hat{U}^{n-1} + \hat{U}^{n+1}\right) + H\hat{P}^n = 0, \ \ n = 1, \cdots, N - 1, \tag{2.14}$$

$$iH\delta_t\hat{V}^n - \frac{k}{2}B\left(\hat{V}^{n-1} + \hat{V}^{n+1}\right) + H\hat{Q}^n = 0, \ \ n = 1, \cdots, N - 1, \tag{2.15}$$

$$iH\delta_t^+\hat{U}^0 - \frac{k}{2}B\left(\hat{U}^0 + \hat{U}^1\right) + H\hat{P}^0 = 0, \tag{2.16}$$

$$iH\delta_t^+\hat{V}^0 - \frac{k}{2}B\left(\hat{V}^0 + \hat{V}^1\right) + H\hat{Q}^0 = 0, \tag{2.17}$$

$$U^0 = \Pi_h\psi, \quad V^0 = \Pi_h\phi, \tag{2.18}$$

$$U^n \in X_h, \quad V^n \in X_h, \quad n = 0, 1, \cdots, N, \tag{2.19}$$

where $P^n, Q^n, \Pi_h\psi, \Pi_h\phi \in X_h$ with

$$P_j^n = \frac{1}{2}\left(|U_j^n|^2 + \beta|V_j^n|^2\right)\left(U_j^{n-1} + U_j^{n+1}\right), \quad j = 1, \cdots, J - 1, \ n = 1, \cdots, N - 1,$$

$$Q_j^n = \frac{1}{2}\left(|V_j^n|^2 + \beta|U_j^n|^2\right)\left(V_j^{n-1} + V_j^{n+1}\right), \quad j = 1, \cdots, J - 1, \ n = 1, \cdots, N - 1,$$

$$P_j^0 = \frac{1}{4}\left(|U_j^0|^2 + |U_j^1|^2 + \beta|V_j^0|^2 + \beta|V_j^1|^2\right)\left(U_j^0 + U_j^1\right), \quad j = 1, \cdots, J - 1,$$

$$Q_j^0 = \frac{1}{4}\left(|V_j^0|^2 + |V_j^1|^2 + \beta|U_j^0|^2 + \beta|U_j^1|^2\right)\left(V_j^0 + V_j^1\right), \quad j = 1, \cdots, J - 1,$$

$$(\Pi_h\psi)_j = \psi(x_j), \quad (\Pi_h\phi)_j = \phi(x_j), \quad j = 1, \cdots, J - 1.$$

The system (2.12)-(2.15) can be rewritten as the following equivalent form

$$i\delta_t\hat{U}^n - \frac{k}{2}AB\left(\hat{U}^{n-1} + \hat{U}^{n+1}\right) + \hat{P}^n = 0, \ \ n = 1, \cdots, N - 1, \tag{2.20}$$

$$i\delta_t\hat{V}^n - \frac{k}{2}AB\left(\hat{V}^{n-1} + \hat{V}^{n+1}\right) + \hat{Q}^n = 0, \ \ n = 1, \cdots, N - 1, \tag{2.21}$$

$$i\delta_t^+\hat{U}^0 - \frac{k}{2}AB\left(\hat{U}^0 + \hat{U}^1\right) + \hat{P}^0 = 0, \tag{2.22}$$

$$i\delta_t^+\hat{V}^0 - \frac{k}{2}AB\left(\hat{V}^0 + \hat{V}^1\right) + \hat{Q}^0 = 0, \tag{2.23}$$

$$U^0 = \Pi_h\psi, \quad V^0 = \Pi_h\phi, \tag{2.24}$$

$$U^n \in X_h, \quad V^n \in X_h, \quad n = 0, 1, \cdots, N, \tag{2.25}$$

where $A = H^{-1}$.

Before we state our main error estimate results, we make the following assumption on the exact solution $u(x,t)$ and $v(x,t)$ of the problem (1.1)-(1.4):

(A) $u, v \in C^4\left([0, T]; W^{0,\infty}(\Omega)\right) \cap C^3\left([0, T]; W^{2,\infty}(\Omega)\right) \cap C^1\left([0, T]; W^{6,\infty}(\Omega) \cap H_0^1(\Omega)\right).$

Define the "error" functions $e^n, \theta^n \in X_h$ as

$$e_j^n = u_j^n - U_j^n, \quad \theta_j^n = v_j^n - V_j^n, \quad j = 0, 1, \cdots, J, \ n = 0, 1, \cdots, N. \qquad (2.26)$$

Then for the compact difference scheme (2.8)-(2.11), we have

**Theorem 2.1.** *Under the assumptions (A), there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, we have the following optimal error estimates for the scheme (2.8)-(2.13)*

$$||e^n|| \lesssim h^4 + \tau^2, \quad ||\theta^n|| \lesssim h^4 + \tau^2, \quad ||\delta_x^+ e^n|| \lesssim h^4 + \tau^2, \quad ||\delta_x^+ \theta^n|| \lesssim h^4 + \tau^2,$$
$$||e^n||_\infty \lesssim h^4 + \tau^2, \quad ||\theta^n||_\infty \lesssim h^4 + \tau^2, \quad n = 1, \cdots, N. \qquad (2.27)$$

## 3. Error Estimate

In this section, we prove that the proposed scheme not only has high order accuracy but also conserves the total masses and energy in the discrete level.

### 3.1. Discrete conservation laws

Corresponding to the conservation laws (1.5),(1.6) and (1.7) preserved by the continuous problem (1.1)-(1.4), the scheme (2.8)-(2.11) conserves the total mass and energy in the discrete level.

**Lemma 3.1.** *The scheme (2.8)-(2.11) satisfies the following conservation laws*

$$M_1^n := ||U^n||^2 \equiv M_1^0, \quad n = 0, 1, \cdots, N, \qquad (3.1)$$
$$M_2^n := ||V^n||^2 \equiv M_2^0, \quad n = 0, 1, \cdots, N, \qquad (3.2)$$
$$\begin{aligned} E^n := &\frac{k}{2}\left( |||\delta_x^+ U^n|||^2 + |||\delta_x^+ U^{n+1}|||^2 + |||\delta_x^+ V^n|||^2 + |||\delta_x^+ V^{n+1}|||^2 \right) \\ &- \frac{1}{2}h\sum_{j=1}^{J-1}\left( |U_j^n|^2|U_j^{n+1}|^2 + |V_j^n|^2|V_j^{n+1}|^2 \right) \\ &- \frac{\beta}{2}h\sum_{j=1}^{J-1}\left( |U_j^n|^2|V_j^{n+1}|^2 + |U_j^{n+1}|^2|V_j^n|^2 \right) \equiv E^0, \quad n = 0, 1, \cdots, N-1. \qquad (3.3) \end{aligned}$$
$$\widetilde{E}^1 := \frac{k}{2}\left( |||\delta_x^+ U^1|||^2 + |||\delta_x^+ V^1|||^2 \right) - \frac{1}{2}h\sum_{j=1}^{J-1}\left( |U_j^1|^4 + |V_j^1|^4 + 2\beta|U_j^1|^2|V_j^1|^2 \right) \equiv \widetilde{E}^0. \quad (3.4)$$

*Here $M_1^n$, $M_2^n$, and $E^n$ are the discrete total "mass" and "energy", respectively.*

*Proof.* Computing the discrete inner product of (2.22) with $\hat{U}^0 + \hat{U}^1$ over $X_h^0$ and then taking the imaginary part, we obtain

$$\frac{1}{\tau}\left( ||U^1||^2 - ||U^0||^2 \right) - \frac{k}{2}\mathrm{Im}\left\langle AB(\hat{U}^0 + \hat{U}^1), \hat{U}^0 + \hat{U}^1 \right\rangle_0 + \mathrm{Im}\left\langle \hat{P}^0, \hat{U}^0 + \hat{U}^1 \right\rangle_0 = 0. \quad (3.5)$$

This, together with

$$\mathrm{Im}\left\langle AB(\hat{U}^0 + \hat{U}^1), \hat{U}^0 + \hat{U}^1 \right\rangle_0 = 0, \quad \mathrm{Im}\left\langle \hat{P}^0, \hat{U}^0 + \hat{U}^1 \right\rangle_0 = 0,$$

gives

$$\frac{1}{\tau}\left(||U^1||^2 - ||U^0||^2\right) = 0,$$

which yields

$$||U^1||^2 = ||U^0||^2. \tag{3.6}$$

Similarly, we obtain

$$||V^1||^2 = ||V^0||^2. \tag{3.7}$$

Computing the discrete inner product of (2.20) with $\hat{U}^{n-1} + \hat{U}^{n+1}$ over $X_h^0$ and then taking the imaginary part, we obtain

$$\frac{1}{2\tau}\left(||U^{n+1}||^2 - ||U^{n-1}||^2\right) - \frac{k}{2}\text{Im}\left\langle AB(\hat{U}^{n-1} + \hat{U}^{n+1}), \hat{U}^{n-1} + \hat{U}^{n+1}\right\rangle_0$$

$$+\text{Im}\left\langle \hat{P}^n, \hat{U}^{n-1} + \hat{U}^{n+1}\right\rangle_0 = 0, \quad n = 1, \cdots, N-1. \tag{3.8}$$

This, together with

$$\text{Im}\left\langle AB(\hat{U}^{n-1} + U^{n+1}), \hat{U}^{n-1} + \hat{U}^{n+1}\right\rangle_0 = 0, \quad \text{Im}\left\langle \hat{P}^n, \hat{U}^{n-1} + \hat{U}^{n+1}\right\rangle_0 = 0,$$

gives

$$\frac{1}{2\tau}\left(||U^{n+1}||^2 - ||U^{n-1}||^2\right) = 0, \quad n = 1, \cdots, N-1. \tag{3.9}$$

This, together with (3.6), gives (3.1). Similarly, we can obtain (3.2).

Computing the discrete inner product of (2.20) with $\hat{U}^{n+1} - \hat{U}^{n-1}$ over $X_h^0$ and then taking the real part, we obtain

$$-\frac{k}{2}\text{Re}\left\langle AB(\hat{U}^{n-1} + \hat{U}^{n+1}), \hat{U}^{n+1} - \hat{U}^{n-1}\right\rangle_0 + \text{Re}\left\langle \hat{P}^n, \hat{U}^{n+1} - \hat{U}^{n-1}\right\rangle_0 = 0. \tag{3.10}$$

Similarly, we obtain

$$-\frac{k}{2}\text{Re}\left\langle AB(\hat{V}^{n-1} + \hat{V}^{n+1}), \hat{V}^{n+1} - \hat{V}^{n-1}\right\rangle_0 + \text{Re}\left\langle \hat{Q}^n, \hat{V}^{n+1} - \hat{V}^{n-1}\right\rangle_0 = 0. \tag{3.11}$$

Adding (3.11) to (3.10), using Lemma 2.3 and noticing that

$$\text{Re}\left\langle \hat{P}^n, \hat{U}^{n+1} - \hat{U}^{n-1}\right\rangle_0 + \text{Re}\left\langle \hat{Q}^n, \hat{V}^{n+1} - \hat{V}^{n-1}\right\rangle_0$$

$$= \frac{1}{2}h\sum_{j=1}^{J-1}|U_j^n|^2\left(|U_j^{n+1}|^2 - |U_j^{n-1}|^2\right) + \frac{\beta}{2}h\sum_{j=1}^{J-1}|V_j^n|^2\left(|U_j^{n+1}|^2 - |U_j^{n-1}|^2\right)$$

$$+\frac{1}{2}h\sum_{j=1}^{J-1}|V_j^n|^2\left(|V_j^{n+1}|^2 - |V_j^{n-1}|^2\right) + \frac{\beta}{2}h\sum_{j=1}^{J-1}|U_j^n|^2\left(|V_j^{n+1}|^2 - |V_j^{n-1}|^2\right)$$

$$= \frac{1}{2}h\sum_{j=1}^{J-1}\left(|U_j^n|^2|U_j^{n+1}|^2 + |V_j^n|^2|V_j^{n+1}|^2 + \beta|V_j^n|^2|U_j^{n+1}|^2 + \beta|U_j^n|^2|V_j^{n+1}|^2\right)$$

$$-\frac{1}{2}h\sum_{j=1}^{J-1}\left(|U_j^{n-1}|^2|U_j^n|^2 + |V_j^{n-1}|^2|V_j^n|^2 + \beta|V_j^{n-1}|^2|U_j^n|^2 + \beta|U_j^{n-1}|^2|V_j^n|^2\right),$$

we can obtain (3.3). Similarly, we can obtain (3.4). $\square$

**Lemma 3.2.** *The difference solution of the scheme* (2.8)-(2.11) *satisfies*

$$||U^n|| \le C, \quad ||V^n|| \le C, \quad ||\delta_x^+ U^n|| \le C, \quad ||\delta_x^+ V^n|| \le C,$$
$$||U^n||_\infty \le C, \quad ||V^n||_\infty \le C, \quad n = 0, 1, \cdots, N. \tag{3.12}$$

*Proof.* It follows from (3.1)-(3.2) that

$$||U^n|| \le C, \quad ||V^n|| \le C, \quad n = 0, 1, \cdots, N. \tag{3.13}$$

Utilizing the Cauchy-Schwarz inequality, discrete Sobolev inequality and the estimate (3.13), we obtain

$$h \sum_{j=1}^{J-1} |U_j^n|^2 |V_j^{n+1}|^2 \le \frac{1}{2} \Big( ||U^n||_4^4 + ||V^{n+1}||_4^4 \Big), \tag{3.14}$$

$$||U^n||_4^4 \le ||U^n||^2 ||U^n||_\infty^2 \le C ||U^n||_\infty^2$$
$$\le \varepsilon ||\delta_x^+ U^n||^2 + C(\varepsilon) ||U^n||^2 \le \varepsilon ||\delta_x^+ U^n||^2 + C, \tag{3.15}$$

where $\varepsilon$ is an arbitrary small number. Noticing (3.14)-(3.15), we obtain from (3.3), (3.4) and (3.13) that

$$||\delta_x^+ U^n|| \le C, \quad ||\delta_x^+ V^n|| \le C, \quad n = 1, \cdots, N. \tag{3.16}$$

By virtue of lemma 2.3, we obtain from (3.13) and (3.16) that

$$||U^n||_\infty \le C, \quad ||V^n||_\infty \le C, \quad n = 1, \cdots, N. \tag{3.17}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 3.2. Error estimate

Define the local truncation error $(\eta^n, \xi^n) \in X_h \times X_h$ of the conservative scheme (2.8)-(2.11) as

$$\mathcal{A}_h \eta_j^n := i \mathcal{A}_h \delta_t u_j^n + \frac{k}{2} \delta_x^2 \left( u_j^{n-1} + u_j^{n+1} \right) + \mathcal{A}_h p_j^n, \tag{3.18}$$

$$\mathcal{A}_h \xi_j^n := i \mathcal{A}_h \delta_t v_j^n + \frac{k}{2} \delta_x^2 \left( v_j^{n-1} + v_j^{n+1} \right) + \mathcal{A}_h q_j^n,$$
$$j = 1, \cdots, J-1, \ n = 1, \cdots, N-1, \tag{3.19}$$

$$\mathcal{A}_h \eta_j^0 := i \mathcal{A}_h \delta_t^+ u_j^0 + \frac{k}{2} \delta_x^2 \left( u_j^0 + u_j^1 \right) + \mathcal{A}_h p_j^0, \quad j = 1, \cdots, J-1, \tag{3.20}$$

$$\mathcal{A}_h \xi_j^0 := i \mathcal{A}_h \delta_t^+ v_j^0 + \frac{k}{2} \delta_x^2 \left( v_j^0 + v_j^1 \right) + \mathcal{A}_h q_j^0, \quad j = 1, \cdots, J-1, \tag{3.21}$$

$$u^0 = \Pi_h \psi, \quad v^0 = \Pi_h \phi, \quad j = 0, 1, \cdots, J, \tag{3.22}$$

$$u^n \in X_J, \quad v^n \in X_J, \quad n = 1, \cdots, N, \tag{3.23}$$

where $p^n, q^n \in X_h$ with

$$p_j^n = \frac{1}{2} \Big( |u_j^n|^2 + \beta |v_j^n|^2 \Big) \Big( u_j^{n-1} + u_j^{n+1} \Big),$$
$$q_j^n = \frac{1}{2} \Big( |v_j^n|^2 + \beta |u_j^n|^2 \Big) \Big( v_j^{n-1} + v_j^{n+1} \Big), \quad j = 1, \cdots, J-1, \ n = 1, \cdots, N-1,$$
$$p_j^0 = \frac{1}{4} \Big( |u_j^0|^2 + |u_j^1|^2 + \beta |v_j^0|^2 + \beta |v_j^1|^2 \Big) \Big( u_j^0 + u_j^1 \Big),$$
$$q_j^0 = \frac{1}{4} \Big( |v_j^0|^2 + |v_j^1|^2 + \beta |u_j^0|^2 + \beta |u_j^1|^2 \Big) \Big( v_j^0 + v_j^1 \Big), \quad j = 1, \cdots, J-1.$$

The system (3.18)-(3.23) can be rewritten as the following equivalent vector form:

$$i\delta_t \hat{u}^n - \frac{k}{2} AB \left( \hat{u}^{n-1} + \hat{u}^{n+1} \right) + \hat{p}^n = \hat{\eta}^n, \quad n = 1, \cdots, N-1, \tag{3.24}$$

$$i\delta_t \hat{v}^n - \frac{k}{2} AB \left( \hat{v}^{n-1} + \hat{v}^{n+1} \right) + \hat{q}^n = \hat{\xi}^n, \quad n = 1, \cdots, N-1, \tag{3.25}$$

$$i\delta_t^+ \hat{u}^0 - \frac{k}{2} AB \left( \hat{u}^0 + \hat{u}^1 \right) + \hat{p}^0 = \hat{\eta}^0, \tag{3.26}$$

$$i\delta_t^+ \hat{v}^0 - \frac{k}{2} AB \left( \hat{v}^0 + \hat{v}^1 \right) + \hat{q}^0 = \hat{\xi}^0, \tag{3.27}$$

$$u^0 = \Pi_h \psi, \quad v^0 = \Pi_h \phi, \tag{3.28}$$

$$u^n \in X_h, \quad v^n \in X_h, \quad n = 0, 1, 2, \cdots, N, \tag{3.29}$$

On the truncations $\eta^n$ and $\xi^n$, using Taylor's expansion, we obtain

**Lemma 3.3.** (*Local truncation error*) *Under assumption (A), there are*

$$||\eta^n||_\infty \lesssim \tau^2 + h^4, \qquad ||\xi^n||_\infty \lesssim \tau^2 + h^4, \qquad n = 0, 1, \cdots, N-1, \tag{3.30}$$

$$||\delta_t \eta^n||_\infty \lesssim \tau^2 + h^4, \qquad ||\delta_t \xi^n||_\infty \lesssim \tau^2 + h^4, \qquad n = 1, \cdots, N-1. \tag{3.31}$$

Based on the lemmas above, we are now ready to prove the main Theorem 2.1.

*Proof.* Theorem 2.1: Subtracting (2.20)-(2.25) from (3.24)-(3.29) gives the following "error" equations:

$$i\delta_t \hat{e}^n - \frac{k}{2} AB \left( \hat{e}^{n-1} + \hat{e}^{n+1} \right) + \hat{\tilde{p}}^n = \hat{\eta}^n, \tag{3.32}$$

$$i\delta_t \hat{\theta}^n - \frac{k}{2} AB \left( \hat{\theta}^{n-1} + \hat{\theta}^{n+1} \right) + \hat{\tilde{q}}^n = \hat{\xi}^n, \quad n = 1, \cdots, N-1, \tag{3.33}$$

$$i\delta_t^+ \hat{e}^0 - \frac{k}{2} AB \left( \hat{e}^0 + \hat{e}^1 \right) + \hat{\tilde{p}}^0 = \hat{\eta}^0, \tag{3.34}$$

$$i\delta_t^+ \hat{\theta}^0 - \frac{k}{2} AB \left( \hat{\theta}^0 + \hat{\theta}^1 \right) + \hat{\tilde{q}}^0 = \hat{\xi}^0, \tag{3.35}$$

$$e^0 = 0, \quad \theta^0 = 0, \tag{3.36}$$

$$e^n \in X_h, \quad \theta^n \in X_h, \quad n = 0, 1, 2, \cdots, N, \tag{3.37}$$

where

$$\hat{\tilde{p}}^n = \hat{p}^n - \hat{P}^n, \quad \hat{\tilde{q}}^n = \hat{q}^n - \hat{Q}^n, \quad n = 0, 1, \cdots, N-1. \tag{3.38}$$

Computing the discrete inner product of (3.34) with $\hat{e}^1$ over $X_h^0$, then taking the imaginary part, we obtain

$$\frac{1}{\tau} ||e^1||^2 + \mathrm{Im} \left\langle \hat{\tilde{p}}^0, \hat{e}^1 \right\rangle_0 = \mathrm{Im} \left\langle \hat{\eta}^0, \hat{e}^1 \right\rangle_0. \tag{3.39}$$

Using Cauchy-Schwarz inequality and Lemma 3.2, we obtain

$$\left| \left\langle \hat{\tilde{p}}^0, \hat{e}^1 \right\rangle_0 \right| \leq C \left( ||e^1||^2 + ||\theta^1||^2 \right), \tag{3.40}$$

$$\left| \left\langle \hat{\eta}^0, \hat{e}^1 \right\rangle_0 \right| \leq ||e^1||^2 + ||\eta^0||^2. \tag{3.41}$$

Plugging (3.40) and (3.41) into (3.39) yields

$$||e^1||^2 \leq C\tau\Big(||e^1||^2 + ||\theta^1||^2\Big) + \tau||\eta^0||^2. \tag{3.42}$$

Similarly, we can obtain

$$||\theta^1||^2 \leq C\tau\Big(||e^1||^2 + ||\theta^1||^2\Big) + \tau||\xi^0||^2. \tag{3.43}$$

Adding (3.43) to (3.42) yields

$$||e^1||^2 + ||\theta^1||^2 \leq C\tau\Big(||e^1||^2 + ||\theta^1||^2\Big) + \tau\Big(||\eta^0||^2 + ||\xi^0||^2\Big). \tag{3.44}$$

If the time step $\tau$ is chosen to be small enough such that $1 - C\tau > 0$, we obtain from (3.44) and Lemma 3.3 that

$$||e^1||^2 + ||\theta^1||^2 \lesssim \tau\Big(\tau^2 + h^4\Big)^2. \tag{3.45}$$

This implies

$$||e^1|| + ||\theta^1|| \lesssim \sqrt{\tau}(\tau^2 + h^4) \lesssim \tau^2 + h^4. \tag{3.46}$$

Computing the discrete inner product of (3.34) with $-\hat{e}^1$ over $X_h^0$, and taking the real part yield

$$\frac{k}{2}|||\delta_x^+ e^1|||^2 + \mathrm{Re}\left\langle \hat{\tilde{p}}^0, \hat{e}^1 \right\rangle_0 = \mathrm{Re}\left\langle \hat{\eta}^0, \hat{e}^1 \right\rangle_0. \tag{3.47}$$

This, together with (3.40), (3.41), (3.46) and Lemma 3.3, gives

$$|||\delta_x^+ e^1|||^2 \lesssim ||e^1||^2 + ||\theta^1||^2 + ||\eta^0||^2 \lesssim \Big(\tau^2 + h^4\Big)^2. \tag{3.48}$$

Similarly, we can obtain

$$|||\delta_x^+ \theta^1|||^2 \lesssim ||e^1||^2 + ||\theta^1||^2 + ||\xi^0||^2 \lesssim \Big(\tau^2 + h^4\Big)^2. \tag{3.49}$$

It follows from (3.48), (3.49) and Lemma 2.2 that

$$||\delta_x^+ \theta^1|| + ||\delta_x^+ e^1|| \lesssim \tau^2 + h^4. \tag{3.50}$$

Computing the discrete inner product of (3.32) with $\hat{e}^{n+1} + \hat{e}^{n-1}$ over $X_h^0$, and taking the imaginary part give

$$\frac{1}{2\tau}\left(||e^{n+1}||^2 - ||e^{n-1}||^2\right) + \mathrm{Im}\left\langle \hat{\tilde{p}}^n, \hat{e}^{n-1} + \hat{e}^{n+1} \right\rangle_0 = \mathrm{Im}\left\langle \hat{\eta}^n, \hat{e}^{n-1} + \hat{e}^{n+1} \right\rangle_0. \tag{3.51}$$

Using Cauchy-Schwarz inequality and Lemma 3.2, we obtain

$$\left|\left\langle \hat{\tilde{p}}^n, \hat{e}^{n-1} + \hat{e}^{n+1} \right\rangle_0\right| \leq C\Big(||e^{n+1}||^2 + ||e^{n-1}||^2 + ||e^n||^2 + ||\theta^n||^2\Big), \tag{3.52}$$

$$\left|\left\langle \hat{\eta}^n, \hat{e}^{n-1} + \hat{e}^{n+1} \right\rangle_0\right| \leq ||e^{n+1}||^2 + ||e^{n-1}||^2 + \frac{1}{2}||\eta^n||^2. \tag{3.53}$$

Plugging (3.52) and (3.53) into (3.51) yields

$$||e^{n+1}||^2 - ||e^{n-1}||^2 \leq C\tau\Big(||e^{n+1}||^2 + ||e^n||^2 + ||e^{n-1}||^2 + ||\theta^n||^2\Big) + \tau||\eta^n||^2. \tag{3.54}$$

Similarly, we obtain

$$||\theta^{n+1}||^2 - ||\theta^{n-1}||^2 \le C\tau\left(||e^n||^2 + ||\theta^{n+1}||^2 + ||\theta^n||^2 + ||\theta^{n-1}||^2\right) + \tau||\xi^n||^2. \quad (3.55)$$

Adding (3.54) to (3.55) and denoting

$$F^n = ||e^n||^2 + ||\theta^n||^2 + ||e^{n+1}||^2 + ||\theta^{n+1}||^2,$$

we obtain

$$F^n - F^{n-1} \le C\tau\left(F^n + F^{n-1}\right) + \tau\left(||\eta^n||^2 + ||\xi^n||^2\right). \quad (3.56)$$

This, together with (3.50), Gronwall's inequality and Lemma 3.3, gives

$$F^n \lesssim \left[\tau^2 + h^4\right]^2. \quad (3.57)$$

This implies

$$||e^n|| + ||\theta^n|| \lesssim \tau^2 + h^4, \quad n = 1, \cdots, N. \quad (3.58)$$

Computing the discrete inner product of (3.32) with $-2\tau\delta_t\hat{e}^n$ over $X_h^0$, and taking the real part lead to

$$\frac{k}{2}\left(|||\delta_x^+ e^{n+1}|||^2 - |||\delta_x^+ e^{n-1}|||^2\right) - 2\tau\mathrm{Re}\left\langle\hat{\tilde{p}}^n, \delta_t\hat{e}^n\right\rangle_0 = -2\tau\mathrm{Re}\left\langle\hat{\eta}^n, \delta_t\hat{e}^n\right\rangle_0, \quad (3.59)$$

where Lemma 2.3 was used. Summing up (3.59) for the superscript $n$ for 1 to $m$ and then replacing $m$ by $n$ yields

$$|||\delta_x^+ e^{n+1}|||^2 + |||\delta_x^+ e^n|||^2$$
$$= |||\delta_x^+ e^1|||^2 + \frac{4}{k}\tau\sum_{l=1}^{n}\mathrm{Re}\left\langle\hat{\tilde{p}}^l, \delta_t\hat{e}^l\right\rangle_0 - \frac{4}{k}\tau\sum_{l=1}^{n}\mathrm{Re}\left\langle\hat{\eta}^l, \delta_t\hat{e}^l\right\rangle_0. \quad (3.60)$$

Noticing that

$$\delta_t\hat{e}^l = -i\frac{k}{2}AB(\hat{e}^{l-1} + \hat{e}^{l+1}) + i\hat{\tilde{p}}^l - i\hat{\eta}^l, \quad (3.61)$$

$$\delta_t\hat{\theta}^l = -i\frac{k}{2}AB(\hat{\theta}^{l-1} + \hat{\theta}^{l+1}) + i\hat{\tilde{q}}^l - i\hat{\xi}^l, \quad (3.62)$$

we have

$$\mathrm{Re}\left\langle\hat{\tilde{p}}^l, \delta_t\hat{e}^l\right\rangle_0 = \mathrm{Re}\left\langle\hat{\tilde{p}}^l, -i\frac{k}{2}AB(\hat{e}^{l-1} + \hat{e}^{l+1}) + i\hat{\tilde{p}}^l - i\hat{\eta}^l\right\rangle_0$$
$$= -\frac{k}{2}\mathrm{Im}\left\langle\hat{\tilde{p}}^l, AB(\hat{e}^{l-1} + \hat{e}^{l+1})\right\rangle_0 - \mathrm{Im}\left\langle\hat{\tilde{p}}^l, \hat{\eta}^l\right\rangle_0, \quad (3.63)$$

where $\mathrm{Im}(f)$ denotes the imaginary part of $f$. Noticing $B = D^2$, $DA = AD$, $\delta_x^+\hat{U}_j^n = \delta_x^+\hat{u}_j^n - \delta_x^+\hat{e}_j^n$, and $||e^n||_\infty \le ||u^n||_\infty + ||U^n||_\infty \le C$, we have

$$\left|\left\langle\hat{\tilde{p}}^l, AB\left(\hat{e}^{l-1} + \hat{e}^{l+1}\right)\right\rangle_0\right| \le \lambda_{H,J-1}^{-1}\left|\left\langle\hat{\tilde{p}}^l, B\left(\hat{e}^{l-1} + \hat{e}^{l+1}\right)\right\rangle_0\right|$$
$$= \lambda_{H,J-1}^{-1}\left|\left\langle\hat{\tilde{p}}^l, \delta_x^2\left(\hat{e}^{l-1} + \hat{e}^{l+1}\right)\right\rangle_0\right| = \lambda_{H,J-1}^{-1}\left|h\sum_{j=0}^{J-1}\delta_x^+\tilde{p}_j^l\delta_x^+\left(\overline{e}_j^{l-1} + \overline{e}_j^{l+1}\right)\right|$$
$$\lesssim ||\delta_x^+ e^{l-1}||^2 + ||\delta_x^+ e^l||^2 + ||\delta_x^+ e^{l+1}||^2 + ||\delta_x^+\theta^l||^2$$
$$\lesssim |||\delta_x^+ e^{l-1}|||^2 + |||\delta_x^+ e^l|||^2 + |||\delta_x^+ e^{l+1}|||^2 + |||\delta_x^+\theta^l|||^2, \quad (3.64)$$

$$\left|\mathrm{Im}\left\langle\hat{\tilde{p}}^l, \hat{\eta}^l\right\rangle_0\right| \lesssim ||e^{l-1}||^2 + ||e^l||^2 + ||e^{l+1}||^2 + ||\theta^l||^2 + ||\eta^l||^2 \lesssim \left(h^4 + \tau^2\right)^2, \quad (3.65)$$

where (3.58), Lemma 2.2, Lemma 2.4, Lemma 3.2 and Lemma 3.3 were used. For the third term of (3.60), we have

$$
\left| \tau \sum_{l=1}^{n} \operatorname{Re} \langle \hat{\eta}^l, \delta_t \hat{e}^l \rangle_0 \right| \leq \left| \tau \sum_{l=1}^{n} \langle \hat{\eta}^l, \delta_t \hat{e}^l \rangle_0 \right|
$$

$$
\leq \frac{1}{4} h \sum_{j=1}^{J-1} \left[ |\eta_j^2|^2 + |\eta_j^{n-1}|^2 + |\eta_j^n|^2 + \tau \sum_{l=2}^{n-1} |\delta_t \eta_j^l|^2 + |e_j^1|^2 + |e_j^n|^2 + |e_j^{n+1}|^2 + \tau \sum_{l=2}^{n-1} |e_j^l|^2 \right]
$$

$$
= \frac{1}{4} \left[ ||\eta^1||^2 + ||\eta^{n-1}||^2 + ||\eta^n||^2 + \tau \sum_{l=2}^{n-1} ||\delta_t \eta^l||^2 + ||e^1||^2 + ||e^n||^2 + ||e^{n+1}||^2 + \tau \sum_{l=2}^{n-1} ||e^l||^2 \right]
$$

$$
\lesssim \left( h^4 + \tau^2 \right)^2, \tag{3.66}
$$

where Lemma 3.3 and (3.58) were used. Plugging (3.63)-(3.66) into (3.60) gives

$$
|||\delta_x^+ e^{n+1}|||^2 \lesssim \left( h^4 + \tau^2 \right)^2 \quad n = 1, \cdots, N-1. \tag{3.67}
$$

This, together with Lemma 2.2, gives

$$
||\delta_x^+ e^n|| \lesssim h^4 + \tau^2, \quad n = 2, \cdots, N. \tag{3.68}
$$

Similarly, we obtain

$$
||\delta_x^+ \theta^n|| \lesssim h^4 + \tau^2, \quad n = 2, \cdots, N. \tag{3.69}
$$

It follows from (3.50), (3.58), (3.69), Lemma 2.4 and Sobolev inequality that

$$
||e^n||_\infty + ||\theta^n||_\infty \lesssim h^4 + \tau^2, \quad n = 1, \cdots, N. \tag{3.70}
$$

This completes the proof.                                                                                                           □

**Remark 3.1.** The compact difference scheme can be used to numerically solve the periodic boundary initial value problem by making some minor changes on the spatial grid (see [22]), and the corresponding error estimate in Theorem 2.1 is also valid.

**Remark 3.2.** Though the proof of the convergence can not be directly extended to the high dimensions, but under some reasonable assumptions on the grid ratio and using the analysis methods used in [2, 3], we can prove that the global error of the compact difference method for the high-dimensional coupled GP equations is still fourth-order in spatial directions and second-order in the temporal direction.

## 4. Approximation Property

In this section, we report numerical results of the compact difference scheme (2.8)-(2.11) of the CNLS equation (1.1)-(1.4) to test the error estimate and the conservation laws.

On the whole real line, the CNLSE has the following exact solution

$$
u(x,t) = \sqrt{\frac{2\alpha}{1+\beta}} \operatorname{sech} \left( \sqrt{2\alpha}(x - \nu t) \right) \exp \left( i\nu x - i \left( \frac{\nu^2}{2} - \alpha \right) t \right), \quad x \in \mathbb{R}, \ t > 0, \tag{4.1}
$$

$$
v(x,t) = -\sqrt{\frac{2\alpha}{1+\beta}} \operatorname{sech} \left( \sqrt{2\alpha}(x - \nu t) \right) \exp \left( i\nu x - i \left( \frac{\nu^2}{2} - \alpha \right) t \right), \quad x \in \mathbb{R}, \ t > 0, \tag{4.2}
$$

where $\alpha, \nu$ are constants. It is easy to see that $u(x,t)$ and $v(x,t)$ decay to zero rapidly as $|x| \to \infty$ for a fixed $t$, so numerically we can solve the CNLS equation in a finite domain $(L_1, L_2)$, where $-L_1, L_2 \gg 1$, i.e., we just only solve the initial-boundary value problem (1.1)-(1.4) with

$$\psi(x) = \sqrt{\frac{2\alpha}{1+\beta}}\mathrm{sech}(\sqrt{2\alpha}x)\exp(i\nu x), \quad x \in [L_1, L_2], \tag{4.3}$$

$$\phi(x) = -\sqrt{\frac{2\alpha}{1+\beta}}\mathrm{sech}(\sqrt{2\alpha}x)\exp(i\nu x), \quad x \in [L_1, L_2]. \tag{4.4}$$

The parameters used in the test are chosen as

$$L_1 = -20, \ \ L_2 = 60, \ \ \alpha = 1, \ \ \nu = 1, \ \ k = 1/2, \ \ \beta = 2/3. \tag{4.5}$$

Denote

$$||Ee(h,\tau)||_\infty = \max_{1 \le n \le N} ||e^n(h,\tau)||_\infty, \quad ||E\theta(h,\tau)||_\infty = \max_{1 \le n \le N} ||\theta^n(h,\tau)||_\infty,$$

$$\mathrm{order1} = \log\left(||Ee(h_1,\tau)||_\infty \,/\, ||Ee(h_2,\tau)||_\infty\right) / \log\left(h_1/h_2\right),$$

$$\mathrm{order2} = \log\left(||E\theta(h_1,\tau)||_\infty \,/\, ||E\theta(h_2,\tau)||_\infty\right) / \log\left(h_1/h_2\right),$$

$$\mathrm{order3} = \log\left(||Ee(h,\tau_1)||_\infty \,/\, ||Ee(h,\tau_2)||_\infty\right) / \log\left(\tau_1/\tau_2\right),$$

$$\mathrm{order4} = \log\left(||E\theta(h,\tau_1)||_\infty \,/\, ||E\theta(h,\tau_2)||_\infty\right) / \log\left(\tau_1/\tau_2\right),$$

where $||e^n(h,\tau)||_\infty$ and $||\theta^n(h,\tau)||_\infty$ denote the maximum norm error of $U$ and $V$, respectively, at $t_n = n\tau$ with the grid size $h$ and time step $\tau$.

Table 4.1: Maximum norm errors of $u$ and $v$ computed by the proposed scheme at $t = 1$ with $\tau = 0.0001$.

| $h$ | $||Ee(h,\tau)||_\infty$ | Order1 | $||E\theta(h,\tau)||_\infty$ | Order2 |
|-----|-----|-----|-----|-----|
| 0.4 | 1.8302e-002 | —— | 1.8306e-002 | —— |
| 0.2 | 1.0608e-003 | 4.11 | 1.0609e-003 | 4.11 |
| 0.1 | 6.4405e-005 | 4.04 | 6.4405e-005 | 4.04 |
| 0.05 | 4.0115e-006 | 4.01 | 4.0115e-006 | 4.01 |

The point-wise errors and convergence order of the proposed scheme, under different values of grid ratios $\lambda = \tau/h^2$, are listed in Table 4.1 and Table 4.2. It is easy to see that the order of convergence in $l^\infty$-norm almost equals to 4 in the spatial direction and 2 in the temporal direction, which supports Theorem 2.1. The values of total masses and energy in the discrete level are listed in Table 4.3. We see from Table 4.3 that the compact difference scheme conserves the discrete masses and energy very well, which supports Lemma 3.1.

Table 4.2: Maximum norm errors of $u$ and $v$ computed by the proposed scheme at $t = 10$ with $h = 0.02$.

| $\tau$ | $\lambda$ | $||Ee(h,\tau)||_\infty$ | Order3 | $||E\theta(h,\tau)||_\infty$ | Order4 |
|-----|-----|-----|-----|-----|-----|
| 0.1 | 250 | 1.40706e-002 | —— | 1.40706e-002 | —— |
| 0.05 | 125 | 3.2237e-003 | 2.13 | 3.2237e-003 | 2.13 |
| 0.025 | 62.5 | 8.1239e-004 | 1.99 | 8.1239e-004 | 1.99 |
| 0.0125 | 31.25 | 2.0456e-005 | 1.99 | 2.0456e-005 | 1.99 |

Table 4.3: Discrete masses and energy computed by the proposed scheme with $h = 0.1, \tau = 0.05$.

| $t_n$ | $Q_1^n$ | $Q_2^n$ | $E^n$ |
|---|---|---|---|
| 0 | 1.697056274847714 | 1.697056274847715 | 0.282824663263791 |
| 2 | 1.697056274847716 | 1.697056274847717 | 0.282824663263803 |
| 4 | 1.697056274847721 | 1.697056274847723 | 0.282824663263802 |
| 6 | 1.697056274847725 | 1.697056274847726 | 0.282824663263814 |
| 8 | 1.697056274847730 | 1.697056274847730 | 0.282824663263815 |
| 10 | 1.697056274847732 | 1.697056274847734 | 0.282824663263821 |

## 5.  Conclusion

In general, in order to establish error estimates of finite difference schemes for nonlinear partial differential equations, one always use the standard energy method on the finite difference schemes. However, due to the difficulty caused by the compact operator $\mathcal{A}_h$, the classical method is no longer valid for analyzing the compact difference scheme. In this paper, in order to overcome the difficulty, we transform the difference scheme into an special and equivalent vector form, then use the energy method on the equivalent system to prove the conserved masses and energy in the discrete level, and we obtain the optimal point-wise error estimate without any restriction on the grid ratio.

Though the compact finite difference scheme can be extended directly to the high-dimensional case, the error estimate is not valid in high dimensions. Because we can't obtain the error bound in the discrete $H^2$-norm, but only in the discrete $H^1$-norm, we know from the embedding theory that it is not enough to get the error bound in the $l^\infty$-norm. The unconditional fourth-order convergence in the discrete $H^1$-norm of finite difference schemes for CNLS equation in 2-D and 3-D is under considering, the future work is to get the error bound in $l^\infty$-norm.

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