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A Stochastic Collocation Method for Delay Differential Equations with Random Input

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Abstract. In this work, we concern with the numerical approach for delay differential equations with random coefficients. We first show that the exact solution of the problem considered admits good regularity in the random space, provided that the given data satisfy some reasonable assumptions. A stochastic collocation method is proposed to approximate the solution in the random space, and we use the Legendre spectral collocation method to solve the resulting deterministic delay differential equations. Convergence property of the proposed method is analyzed. It is shown that the numerical method yields the familiar exponential order of convergence in both the random space and the time space. Numerical examples are given to illustrate the theoretical results.

AMS subject classifications: 65C20, 65M70

Key words: Delay differential equations, stochastic collocation, sparse grid, legendre spectral method.

1 Introduction

Using delay differential equations (DDEs) to model biological/engineer systems has a long history, dating to Malthus, Verhulst, Lotka and Volterra. Recently, DDE models have been arisen in many diverse applications including infectious disease dynamics including primary infection [11], immune response [24], tumor growth [23] and neural networks [4], to name a few. As the primary goal for using these models is to better our understanding of real word phenomena, it is becoming clear that the simple models can not capture the whole dynamics observed in natural systems. Thus, in real applications, the systems used are usually build up by a large number of DDEs with a lot of given data.

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The linear systems of DDEs admit the following form

$$\mathbf{u}'(t)=\mathbf{A}(t)u(t)+\mathbf{B}(t)\mathbf{u}(t-q(t)),\quad t\in I:=[0,T], \tag{1.1}$$

where $\mathbf{u}=(u_1,\cdots,u_N)^T$ is a vector of N unknown functions, and $\mathbf{A}$, $\mathbf{B}$ are coefficient matrices in $\mathbb{R}^{N\times N}$. The delay function $q(t)$ is assumed to satisfy

$$0<q(t)<t,\quad t\in(0,T],\quad q(0)=q\geq 0.$$

There are various types of delays in real applications [3], and here we will consider two widely used types of them, namely,

$$\text{Constant delay}: \qquad q(t)=\tau>0, \tag{1.2a}$$
$$\text{Pantograph delay}: \qquad t-q(t)=qt,\quad 0<q<1. \tag{1.2b}$$

The following initial conditions are needed for problem (1.1)

$$\mathbf{u}(t)=\mathbf{u}_0,\quad t\in[-\tau,0], \qquad \text{for constant delay}, \tag{1.3a}$$
$$\mathbf{u}(0)=\mathbf{u}_0, \qquad \text{for pantograph delay}. \tag{1.3b}$$

Generally speaking, the value of interest $\mathbf{u}$ is computed based on the input data $(\mathbf{A}(t),\mathbf{B}(t),\mathbf{u}_0)$ which are provided mainly by experimental measurements or a priori knowledge. It turns out that in many practical applications the input data are not known precisely a priori, which due to error in experiments and/or less of knowledge, namely, uncertainty in the given data. A popular way to deal with such an issue is to model these uncertain data as random variables/random functions [8]. For the context of delay differential equations, this will introduces the following random/parametric delay differential equations

$$\begin{cases} \mathbf{u}'(t,\vec{\xi})=\mathbf{A}(t,\vec{\xi})\mathbf{u}(t,\vec{\xi})+\mathbf{B}(t,\vec{\xi})\mathbf{u}(t-q(t),\vec{\xi}),\quad t\in I:=[0,T],\\ \mathbf{u}(t)=\mathbf{u}_0,\quad t\in[-q,0],\quad \text{for constant delay},\\ \mathbf{u}(0)=\mathbf{u}_0,\quad \text{for pantograph delay}, \end{cases} \tag{1.4}$$

where $\vec{\xi}=(\xi_1,\cdots,\xi_M)$ is a random vector of M random parameters. One usually assumes that the random parameters are independent with each other, and further more, there is a corresponding probability density function $\rho(\xi_i)$ for each random parameter ξ_i in its supporting domain $[a_i,b_i]$.

Such a framework for problems with uncertain input has been widely used by researchers for partial/ordinary differential equations (PDE/ODE) models [8, 28, 29], and the resulting problems are also known as stochastic PDEs/ODEs. Stochastic modeling approaches for such problems can be categorized as either non-intrusive or intrusive. Intrusive approaches, such as generalized polynomial chaos methods (see, e.g., [8, 28] and references therein), usually result in deterministic coupled systems, and this require the

update of existing codes or the development of a new code. More over, the resulting system of equations can be larger than the original, deterministic system.

Non-intrusive approaches, such us Mento Carlo methods [6], the least-squares approach [7,12,30], and stochastic collocation (SC) methods [13,14,26,27], have the advantage that the deterministic version of a model can be used to generate an ensemble of sample solutions from which the statistics can be calculated. The disadvantage is that a large number of samples are often required, leading to large computational costs, especially for problems with a large number of random variables. The full comparison between non-intrusive and intrusive methods is problem dependent and remains a active research topic [2]. We remark that both the intrusive gPC methods and the non-intrusive SC techniques can exploit the possible regularity of the solution with respect to the random parameters to achieve faster convergence [9,10,21,31–33].

In this work, we concern with the sparse grid stochastic collocation approach for problem (1.4) in the random space, and this is motivated by the regularity analysis which will be given in Section 3. As we will see, the solution of problem considered admits very good regularity properties in the random space, under reasonable assumptions on the given data. This motivate us to use the high order SC methods. For each selected collocation points, the problem becomes a system of deterministic delay differential equations. We then introduce the Legendre spectral collocation methods [1] to solve the resulting deterministic DDEs. Convergence properties of the proposed method is analyzed. It is shown that the methods yields the familiar exponential order of convergence in both the random spaces and the time space. Numerical examples will be provided to verify the efficiency of the numerical methods and the theoretical findings.

The rest of the paper is organized as follows. In Section 2 we introduce some basic theoretical results for problem (1.4), particularly, via a single DDE, we shall show that the solution admits good regularity with respect to the random parameters. The combination of the stochastic collocation method and Legendre spectral methods for DDEs will be discussed in section 3, and this is followed by some numerical examples in Section 4. We finally give some conclusion remarks in Section 5.

2 Basic setting and regularity properties

Without loss of generality, we assume that each random parameter ξ_i in equation (1.4) is located in $\Gamma_i=[-1,1]$, and denote with $\Gamma=\Pi_{i=1}^{M}\Gamma_i$. It is assumed that the given functions in (1.4) are smooth on their respective domain $[0,T]$. This implies that for pantograph-type DDEs the solutions are (globally) smooth on I for any fixed parameter $\vec{\xi}\in\Gamma$ [3]. While for constant-type DDEs, the solution can admit discontinuous even if the given data are smooth, however, in this paper, we only consider constant-type DDEs with smooth solutions in the time space. We leave problems with discontinuous for future study. Thus, it appears natural to employ spectral methods for the numerical solution of these functional equations on I since the resulting spectral approximations are globally smooth, too, and

in addition exhibit high-order convergence. For more theoretical results on deterministic delay differential equations, please refer to [3]. In the following, we will restrict our discussion on a single stochastic DDE (2.1), and we will show that the corresponding exact solution admits good regularity properties with respect to the random parameters, provided that the given data satisfy reasonable assumptions.

$$\begin{cases} u'(t,\vec{\xi})=a(t,\vec{\xi})u(t,\vec{\xi})+b(t,\vec{\xi})u(t-q(t),\vec{\xi}), \quad t\in I:=[0,T], \\ u(t)=u_0(t,\vec{\xi}), \quad t\in[-\tau,0], \quad \text{for constant delay}, \\ u(0)=u_0(\vec{\xi}), \quad \text{for pantograph delay}. \end{cases} \tag{2.1}$$

For systems of stochastic DDEs, the proof is quite similar. For simplicity, we also assume that the random variables are uniformly distributed so that $\rho=1$. We first give the following Lemma.

Lemma 2.1. *Consider problem* (2.1)*, assume that the given data are sufficient smooth in the time space I, and satisfy*

$$\max_{\Gamma\otimes I}|a(t,\vec{\xi})|<+\infty, \quad \max_{\Gamma\otimes I}|b(t,\vec{\xi})|<+\infty, \quad \max_{\Gamma}|u_0|<+\infty. \tag{2.2}$$

Then, we have

$$\Big(\mathbb{E}\big[\|u\|^2_{L^2(I)}\big]\Big)^{1/2}=\Big(\int_\Gamma\int_I u^2(t,\vec{\xi})dtd\vec{\xi}\Big)^{1/2}\lesssim\max_{\Gamma\otimes I}|u(t,\vec{\xi})|<C(I)<+\infty, \tag{2.3}$$

where $\mathbb{E}[\cdot]$ stands for the expectation and $C(I)$ is a constant which depends on T and the given data.

Proof. By (2.1), we have

$$u(t,\vec{\xi})=\int_0^t a(t,\vec{\xi})u(t,\vec{\xi})dt+\int_0^t b(t,\vec{\xi})u(t-q(t),\vec{\xi})dt+u_0(0,\vec{\xi}). \tag{2.4}$$

For constant-type delay ($q(t)=\tau$), we have

$$\begin{aligned} |u(t,\vec{\xi})| &\le \max_{\Gamma\otimes I}|a(t,\vec{\xi})|\int_0^t|u(t,\vec{\xi})|dt+\max_{\Gamma\otimes I}|b(t,\vec{\xi})|\int_{-\tau}^{t-\tau}|u(s,\vec{\xi})|ds+\max\Gamma|u_0(0,\vec{\xi})| \\ &\le \max_{\Gamma\otimes I}|a(t,\vec{\xi})|\int_0^t|u(t,\vec{\xi})|dt+\max_{\Gamma\otimes I}|b(t,\vec{\xi})|\int_{-\tau}^{0}|u_0(s,\vec{\xi})|ds+\max_{\Gamma\otimes I}|b(t,\vec{\xi})|\int_0^{t-\tau}|u(s,\vec{\xi})|ds \\ &\le \Big(\max_{\Gamma\otimes I}|a(t,\vec{\xi})|+\max_{\Gamma\otimes I}|b(t,\vec{\xi})|\Big)\int_0^t|u(t,\vec{\xi})|dt+\max_{\Gamma\otimes I}|b(t,\vec{\xi})|\int_{-\tau}^{0}|u_0(s,\vec{\xi})|ds, \end{aligned} \tag{2.5}$$

then, the desired result (2.3) is obtained by using together the above inequality and the Gronwall inequality.

For pantograph-type delay ($t-q(t)=qt$), we have

$$\begin{aligned}|u(t,\vec{\xi})| &\leq \max_{\Gamma\otimes I}|a|\int_0^t |u(t,\vec{\xi})|dt+\max_{\Gamma\otimes I}|b|\int_0^t |u(qt,\vec{\xi})|dt+\max_{\Gamma}|u_0| \\ &\leq \max_{\Gamma\otimes I}|a|\int_0^t |u(t,\vec{\xi})|dt+\frac{1}{q}\max_{\Gamma\otimes I}|b|\int_0^{qt} |u(s,\vec{\xi})|ds+\max_{\Gamma}|u_0| \\ &\leq \Big(\max_{\Gamma\otimes I}|a|+\frac{1}{q}\max_{\Gamma\otimes I}|b|\Big)\int_0^t |u(t,\vec{\xi})|dt+\max_{\Gamma}|u_0|, \end{aligned} \tag{2.6}$$

again, the desired result (2.3) can be obtained by using together the above inequality and the Gronwall inequality, and this complete the proof. □

Notice that the proof for the constant-type delay and the pantograph-type delay is quite similar, therefore, in what follows, we will only give the regularity analysis for the pantograph-type delay problem. The proof for constant-type delay problems can be done in a similar way as in Lemma 2.1.

2.1 Regularity of $\partial_{\xi_i}u$

We are now ready to provide the regularity results for $\partial_{\xi_i}u$, $\forall 1\leq i\leq M$.

Theorem 2.1. *Consider problem* (2.1) *with pantograph-type delay, if the conditions in Lemma 2.1 hold, and furthermore, the given data satisfy*

$$\max_{\Gamma\otimes I}|\partial_{\xi_i}a|<+\infty,\quad \max_{\Gamma\otimes I}|\partial_{\xi_i}b|<+\infty,\quad \max_{\Gamma}|\partial_{\xi_i}u_0|<+\infty. \tag{2.7}$$

Then, we have

$$\Big(\mathbb{E}\big[\|\partial_{\xi_i}u\|^2_{L^2(I)}\big]\Big)^{1/2}=\Big(\int_\Gamma\int_I(\partial_{\xi_i}u)^2dtd\vec{\xi}\Big)^{1/2}\lesssim\max_{\Gamma\otimes I}|\partial_{\xi_i}u|<C_i(I)<+\infty, \tag{2.8}$$

where $C_i(I)$ is a constant depends on T and the given data.

Proof. Differential the stochastic DDE in both sides with respect to ξ_i and let $v=\partial_{\xi_i}u$, we obtain

$$\begin{aligned}v(t,\vec{\xi}) &= \int_0^t a(t,\vec{\xi})v(t,\vec{\xi})dt+\int_0^t b(t,\vec{\xi})v(qt,\vec{\xi})dt \\ &\quad+\int_0^t\big(\partial_{\xi_i}a\big)u(t,\vec{\xi})dt+\int_0^t\big(\partial_{\xi_i}b\big)u(qt,\vec{\xi})dt+\partial_{\xi_i}u_0(\vec{\xi}).\end{aligned} \tag{2.9}$$

Using the similar idea as in Lemma 2.1, we get

$$\begin{aligned}|v(t,\vec{\xi})| &\leq \Big(\max_{\Gamma\otimes I}|a|+\frac{1}{q}\max_{\Gamma\otimes I}|b|\Big)\int_0^t|v(t,\vec{\xi})|dt \\ &\quad+\Big(\max_{\Gamma\otimes I}|\partial_{\xi_i}a|+\frac{1}{q}\max_{\Gamma\otimes I}|\partial_{\xi_i}b|\Big)\int_0^t|u(t,\vec{\xi})|dt+|\partial_{\xi_i}u_0(\vec{\xi})|.\end{aligned} \tag{2.10}$$

Then, the desired result (2.8) is obtained by using together the above inequality, Lemma 2.1, assumption (2.7), and the Gronwall inequality. □

2.2 Regularity of $\partial^2_{\xi_i\xi_j} u$

By Lemma 2.1 and Theorem 2.1, we can derive the regularity properties for $\partial^2_{\xi_i\xi_j} u$.

Theorem 2.2. *Consider problem* (2.1) *with pantograph-type delay, if the conditions in Lemma 2.1 and Theorem 2.1 hold and further more, the given data satisfy*

$$\max_{\Gamma\otimes I}|\partial^2_{\xi_i\xi_j} a|<+\infty,\quad \max_{\Gamma\otimes I}|\partial^2_{\xi_i\xi_j} b|<+\infty,\quad \max_{\Gamma}|\partial^2_{\xi_i\xi_j} u_0|<+\infty. \tag{2.11}$$

Then, we have

$$\left(\mathbb{E}\left[\|\partial^2_{\xi_i\xi_j} u\|^2_{L^2(I)}\right]\right)^{1/2}=\left(\int_\Gamma\int_I(\partial^2_{\xi_i\xi_j} u)^2 dt d\vec{\xi}\right)^{1/2}\lesssim\max_{\Gamma\otimes I}|\partial^2_{\xi_i\xi_j} u|<C_{ij}(I)<+\infty, \tag{2.12}$$

where $C_{ij}(I)$ is a constant depends on T and the given data.

Proof. Differential twice the stochastic DDE in both sides with respect to ξ_i and ξ_j and let $w=\partial^2_{\xi_i\xi_j} u$, we obtain

$$\begin{aligned}
w(t,\vec{\xi})=&\int_0^t a(t,\vec{\xi})w(t,\vec{\xi})dt+\int_0^t b(t,\vec{\xi})w(qt,\vec{\xi})dt\\
&+\int_0^t\left(\partial_{\xi_i}a\right)v_j(t,\vec{\xi})dt+\int_0^t\left(\partial_{\xi_i}b\right)v_j(qt,\vec{\xi})dt\\
&+\int_0^t\left(\partial_{\xi_j}a\right)v_i(t,\vec{\xi})dt+\int_0^t\left(\partial_{\xi_j}b\right)v_i(qt,\vec{\xi})dt\\
&+\int_0^t\left(\partial^2_{\xi_i\xi_j}a\right)u(t,\vec{\xi})dt+\int_0^t\left(\partial^2_{\xi_i\xi_j}b\right)u(qt,\vec{\xi})dt+\partial^2_{\xi_i\xi_j}u_0(\vec{\xi}).
\end{aligned}\tag{2.13}$$

Using the similar idea as in Theorem 2.1, we get

$$\begin{aligned}
|w(t,\vec{\xi})|\le&\left(\max_{\Gamma\otimes I}|a|+\frac{1}{q}\max_{\Gamma\otimes I}|b|\right)\int_0^t|w(t,\vec{\xi})|dt\\
&+\left(\max_{\Gamma\otimes I}|\partial_{\xi_i}a|+\frac{1}{q}\max_{\Gamma\otimes I}|\partial_{\xi_i}b|\right)\int_0^t|v_j(t,\vec{\xi})|dt\\
&+\left(\max_{\Gamma\otimes I}|\partial_{\xi_j}a|+\frac{1}{q}\max_{\Gamma\otimes I}|\partial_{\xi_j}b|\right)\int_0^t|v_i(t,\vec{\xi})|dt+\left|\partial^2_{\xi_i\xi_j}u_0(\vec{\xi})\right|.
\end{aligned}\tag{2.14}$$

Then, the desired result (2.12) is obtained by using together Theorem 2.1, Lemma 2.1 and the Gronwall inequality. □

Remark 2.1. It is clear that if the given data satisfy some further assumptions, the solution of the problem should have higher regularity, namely, we can get the regularity properties for $\partial^3_{\xi_i\xi_j\xi_k}$ and so on. A more detailed set of conditions can be found following the above procedures, which will be omitted in this paper. We also remark that for certain

cases, the solution of the DDEs may have uniformly bounded derivative (in the random space) for any order, and this is in deed the case for many stochastic PDE models [9,10,32]. In such cases, one can expect the so called "analytic regularity" of the solution with respect to the random parameters. This will results in the spectral convergence when one use the polynomial based approaches in the random space, such as gPC methods and SC methods.

3 Numerical methods

In this section, we discuss the numerical approach for problem (2.1). The numerical scheme consists of a sparse grid stochastic collocation method in the random space and the Legendre spectral collocation method with respect to t.

3.1 A stochastic collocation methods in parametric spaces

In stochastic collocation method, one first choose a set of collocation points in the random space $\{\vec{\xi}_k\}_{k=1}^{\Theta} \in \Gamma$. Then, we compute the solution $u(t,\vec{\xi}_k)$ for each collocation point. Finally, we build a global polynomial approximation upon these evaluations

$$u^{\Theta}(t,\vec{\xi}) = \sum_{k=1}^{\Theta} u_k(t,\vec{\xi}_k) L_k(\vec{\xi}), \tag{3.1}$$

with suitable multivariate polynomials $\{L_k\}_{k=1}^{\Theta}$ such as Lagrange polynomials.

It is clear that the computational complexity of the stochastic collocation methods is Θ times that of a deterministic problem, where Θ is the total number of collocation points. As the main focus here is on multidimensional random spaces, thus, it is important to choose a nodal set Θ with fewest possible number of points under a prescribed accuracy requirement. In the following, we will present several choices of such collocation points.

A nature choice is the tensor product of one-dimensional Lagrange interpolation polynomials, precisely, let $\mathcal{U}^i$ be the one dimensional interpolation formula in the ith direction, namely,

$$\mathcal{U}^i(u) = \sum_{j=1}^{\Theta^i} u(\xi_j^i) \cdot a_j^i, \quad j = 1, \cdots, \Theta^i,$$

where $\{\xi_j^i\}_{j=1}^{\Theta^i}$ are the Θ^i collocation points in the direction ξ^i, and $\{a_j^i\}_{j=1}^{\Theta^i}$ are the corresponding weights. In the multivariate case $M>1$, the tensor product formulas are

$$\mathcal{I}(u) \equiv \left(\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_M}\right)(u) = \sum_{k_1=1}^{\Theta^{i_1}} \cdots \sum_{k_M=1}^{\Theta^{i_M}} \left(a_{k_1}^{i_1} \cdots a_{k_M}^{i_M}\right) u\left(\xi_{k_1}^{i_1}, \cdots, \xi_{k_M}^{i_M}\right).$$

Clearly, the above product formula needs $\Theta=\Theta^{i_1}\cdots\Theta^{i_M}$ nodal points. If we choose to use the same interpolating function in each dimension with the same number of points, i.e., $\Theta^{i_1}=\Theta^{i_2}=\cdots=\Theta^{i_M}\equiv m$, the total number of points is $\Theta=m^M$, which grows quickly in high dimensions.

A more reasonable choice is the Smolyak formulas [20]. The Smolyak algorithm is a linear combination of product formulas, and the linear combination is chosen in such a way that the interpolation property for $M=1$ is preserved for $M>1$. Compared to the tensor product rule, only products with a relatively small number of points are used and the resulting nodal set has significantly less number of nodes Much research has been devoted to the Smolyak algorithm since its introduction in [20], see, e.g., [15,16]. Starting with the one-dimensional interpolation formula, the Smolyak algorithm is given by

$$A(q,M)=\sum_{q-M+1\leq|\mathbf{i}|\leq q}(-1)^{q-|\mathbf{i}|}\binom{M-1}{q-|\mathbf{i}|}\left(\mathcal{U}^{i_1}\otimes\cdots\otimes\mathcal{U}^{i_M}\right),$$

where $\mathbf{i}=(i_1,\cdots,i_M)$, and the index q is called *level*. To compute $A(q,M)$, we only need to evaluate function on the *sparse grids*

$$\Theta_q=\mathcal{H}(q,M)=\bigcup_{q-M+1\leq|\mathbf{i}|\leq q}\left(\Theta^{i_1}\otimes\cdots\otimes\Theta^{i_M}\right).$$

In the following, we propose two different abscissas in the construction of the Smolyak formula.

A popular choice is the **Clenshaw-Curtis** abscissas which is introduced in [5]. These abscissas are the extrema of Chebyshev polynomials and for $\Theta^i>1$, we choose

$$\xi_k^i=-\cos\left(\frac{\pi(j-1)}{k-1}\right),\quad k=1,\cdots,\Theta^i.$$

More over, we set $\xi_1^i=0$ if $\Theta^1=1$ and further require the number of abscissas Θ^i in each level to grow according to the following formula

$$\Theta^i=2^{i-1}+1,\quad\forall i>1.$$

Using such particular choice, we can obtain nested sets of abscissas, i.e., $\Theta^i\in\Theta^{i+1}$ and thereby $\Theta_q\in\Theta_{q+1}$.

Another frequently used abscissas is the **Gaussian abscissas**, in which one uses the zeros of the orthogonal polynomials with respect to some positive weight. But, these Gaussian abscissas are in general not nested, as mentioned above for Clenshaw-Curtis points. Nevertheless, as in the Clenshaw-Curtis case, we choose the number Θ^i of abscissas that are used by $\mathcal{U}^i$. One can refer [22] for an insightful comparison of quadrature formulas based on Clenshaw-Curtis and Gaussian abscissas. The natural choice of the weight should be the probability density function ρ of the random variables ξ^i. Examples of isotropic sparse grids, constructed from the nested Clenshaw-Curtis abscissas and

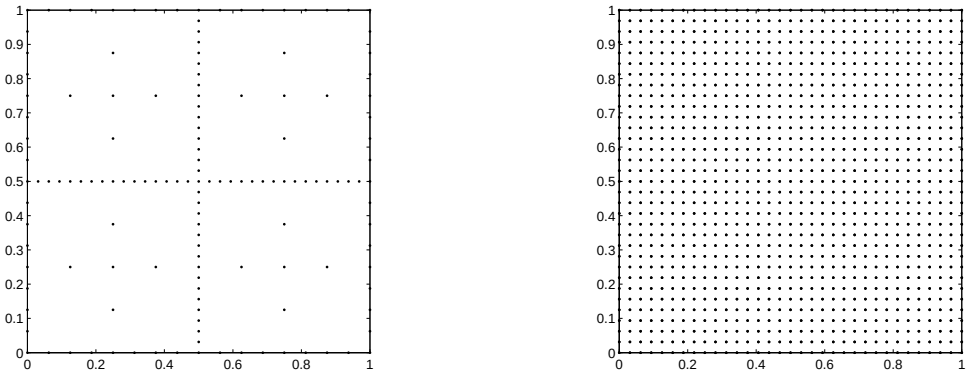


Figure 1: Sparse Grid v.s. Tansor Product: Level=5"clenshaw-curtis".

the non-nested Gaussian abscissas are shown in Fig. 1, where we have considered a two-dimensional parameter space and a maximum level $level=5$ (sparse grid $\mathcal{H}(5,2)$). It is clear to see that the sparse grid setting uses smaller number of collocation points compared to the tensor product.

3.2 Deterministic solver

By using the SC approach in the random space, one arrives at a set of uncoupled deterministic DDEs (DDEs for different points are independent, while each DDEs can still be coupled if the original DDEs are coupled). In this section, we introduce the Legendre spectral collocation method to solve the resulting deterministic DDEs. Such a numerical method has been introduced for example in [1]. The following discussion is concerned with pantograph-type DDEs. However, the corresponding numerical method for constant-type DDEs can be done in the same way [25]. For other numerical methods on DDEs, please refer to [3] and references therein.

Consider the following one dimensional deterministic DDE

$$\begin{cases} u'(t)=a(t)u(t)+b(t)u(qt), & t\in I:=[0,T], \\ u(0)=u_0. \end{cases} \tag{3.2}$$

Let $\{t_k\}_{k=0}^N$ be the set of the $(N+1)$ Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points in $[-1,1]$ and denote by $\mathcal{P}_N$ the space of real polynomials of degree not exceeding N.

Integration of (3.2) from $[0,t_i]$ leads to

$$u(t_i)=u_0+\int_0^{t_i}a(s)u(s)ds+\int_0^{t_i}b(s)u(qs)ds. \tag{3.3}$$

Let us employ the linear transformation

$$s_\theta^i=\frac{t_i}{2}\theta+\frac{t_i}{2},$$

and let $\{\omega_k\}_{k=0}^N$ be the corresponding weights. Suppose that the spectral approximation $U(t)$ has the form

$$u(t)\approx U(t):=\sum_{j=0}^{N}u(t_j)F_j(t),\quad t\in[-1,1],$$

where the $F_j(t)$ are the Lagrange interpolation polynomials with respect to the set $\{\theta_k\}_{k=0}^N$, the spectral approximation to the transformed equation (3.3) yields,

$$u(t_i)=u_0+\frac{t_i}{2}\int_{-1}^{1}a(s_\theta^i)u(s_\theta^i)d\theta+\frac{t_i}{2}\int_{-1}^{1}b(s_\theta^i)u(qs_\theta^i)d\theta. \tag{3.4}$$

Then, the approximated solution is determined by the spectral equations

$$U(t_i)=u_0+\frac{t_i}{2}\sum_{k=0}^{N}a(s_{ik})U(s_{ik})w_k+\frac{t_i}{2}\sum_{k=0}^{N}b(s_{ik})U(qs_{ik})w_k, \tag{3.5}$$

where

$$U(s_{ik}):=\sum_{j=0}^{N}u(t_j)F_j(s_{ik}),\quad s_{ik}:=s_{\theta_k}^i=\frac{t_i}{2}(\theta_k+1).$$

Setting $U_N:=[U(t_0),\cdots,U(t_N)]^T$, $b:=[u_0,\cdots,u_0]^T$ and

$$A_{ij}:=\frac{t_i}{2}\sum_{k=0}^{N}(a(s_{ik})F_j(s_{ik})+b_1(s_{ik})F_j(q_1s_{ik}),$$

$(i,j=0,1,\cdots,N)$, we can write the spectral equations (3.5) in the form

$$(I-A_N)U_N=b, \tag{3.6}$$

where $A_N:=[A_{ij}]\in\mathcal{R}^{(N+1)\times(N+1)}$.

Remark 3.1. The entire numerical methods is: for every selected simple $\vec{\xi}_k$, we use the above numerical method to get a approximate solution $\tilde{u}(t,\vec{\xi}_k)$, then, the numerical solution will be given by

$$\tilde{u}^\Theta(t,\vec{\xi})=\sum_{k=1}^{\Theta}\tilde{u}(t,\vec{\xi}_k)L_k(\vec{\xi}). \tag{3.7}$$

3.3 Error separation

Note the total error $\|u-\tilde{u}^{\Theta}\|$ in the space $L^2(T)\otimes L^2(\Gamma)$ can be splitted into the following two parts

$$\epsilon=\|u-\tilde{u}^{\Theta}\|\leq\|u-u^{\Theta}\|+\|u^{\Theta}-\tilde{u}^{\Theta}\|=\epsilon_I+\epsilon_{II}. \tag{3.8}$$

The first term is the interpolation error introduced by the SC methods in the random space, and such error has been well investigated be many researchers [13,14], typically, if the interpolated functions have bounded S-order mix derivatives, then the interpolation error behaves like [13]

$$\epsilon_I\leq C(w+1)^{2M}2^{-S(w+1)}\max_{0\leq k_1,\cdots,k_M\leq S}\|\partial^{k_1}_{\xi_1}\cdots\partial^{k_M}_{\xi_M}u\|_{L^2(\Gamma,L^2(D))},$$

where w is the *level* in sparse grid methods. The second term is introduced by the deterministic solver, and the error can be estimated via the following theorem [1]:

Theorem 3.1. *Consider the spectral approximation method for DDEs in Section 3.2, If the gien data are sufficiently smooth, then the error between the exact solution u and numerical solution U behaves as*

$$\|u-U\|_{L^1(I)}\leq CN^{-m-\frac{1}{2}}|b(\bullet)u(q\bullet+q-1)|_{\bar{H}^m_N(I)}+CN^{\frac{1}{2}-m}|b|_{\bar{H}^m_N(I)}\|u\|_{L^2(I)}.$$

Here m indicates the regularity of the solution with respect to the time variable. One can find more details for the definition for corresponding norm $|\cdot|_{\bar{H}^m_N(I)}$ in [1]. By the above results, we can expect the spectral convergence of the entire numerical approach.

4 Numerical examples

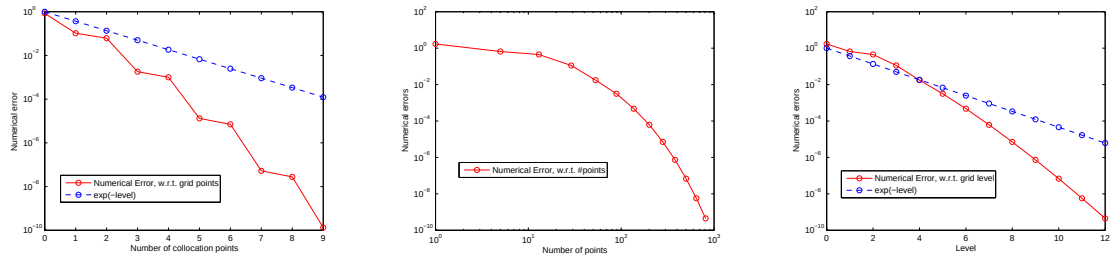
In this section, we provided with several numerical examples. In the computations, to avoid the numerical error introduced by time discretizations, we will use 15 Legendre points in the deterministic DDE solver. Gaussian abscissas sparse grid is used in the approximation in the random spaces. The following mean square error is used to evaluate the numerical scheme

$$\epsilon=\Big(\int_\Gamma\int_I\big(u^{exact}-u^{num}\big)^2dtd\vec{\xi}\Big)^{1/2}.$$

Let us first consider the following problems

$$\text{I:}\begin{cases}u'(t)=u\Big(t-\dfrac{\pi}{2}\Big)+g(\alpha,t), & 0\leq t\leq\dfrac{\pi}{2},\\ u(t)=\sin(\alpha t), & -\dfrac{\pi}{2}\leq t\leq 0,\end{cases}\qquad\text{II:}\begin{cases}u'(t)=\cos(\alpha t)u(qt)+g(t,\beta,\alpha),\\ u(t=0)=0.\end{cases}$$

Problem I is a delay problem with constant delay, and we chose a special g such that the exact solution for this problem is $u=\sin(\alpha t)$, where α is a uniformly distributed random



(a) (b) (c)

Figure 2: (a) Example I, mean square error v.r.t. number of collocation points. Example II, mean square error v.r.t. number of collocation points (b) and mean square error v.r.t sparse grid *level* (c).

variable in $[-1,1]$. This is a one dimensional stochastic problem. Problem II is a delay problem with Pantograph-type delay, let

$$g(t,\beta,\alpha)=-\Big(\alpha\sin(\alpha t)\exp^{-\beta t}+\alpha\beta\cos(\alpha t)\exp^{-\beta t}+\cos(\alpha qt)\exp^{-\beta qt}\Big),$$

then, it can be checked that the exact solution is $u=\cos(\alpha t)\exp^{-\beta t}$. Again, α,β are traded as two independent random variables which are uniformly distributed in the domain $[-1,1]$. Thus, we will use two dimensional sparse grid collocation methods to approximate the solution in the random spaces.

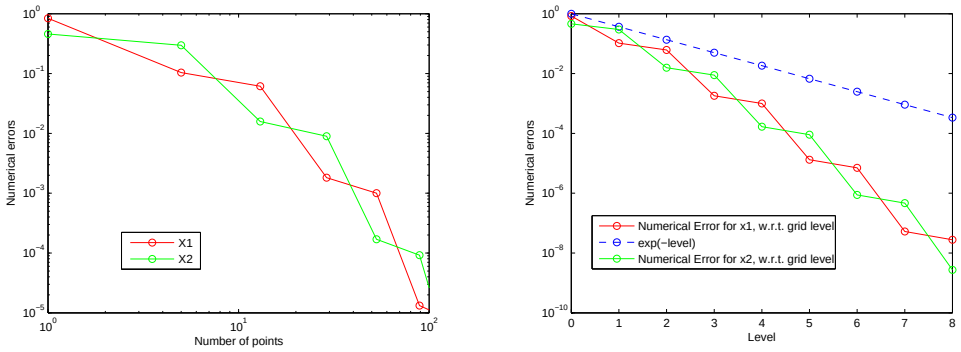
In Fig. 2(a), we plot the mean square error with respect to the collocation points for Example I, where spectral convergence is obtained. In Figs. 2(b) and (c), convergence properties for Example II are provided, the left plot is the mean square error with respect to the collocation points and the right plot is the mean square error with respect to the sparse grid *level*. It can be seen that the numerical methods convergent very fast.

Now, we consider the following delay systems

$$\text{III}:\begin{cases} x_1'(t)=\sin(\alpha t)x_1(qt)-\cos(t)x_2(qt)+g_1(t),\\ x_2'(t)=\cos(\beta t)x_1(qt)+\sin(t)x_2'(qt)+g_2(t),\\ x_1(-1)=a_1,\quad x_2(-1)=b_1,\end{cases}$$

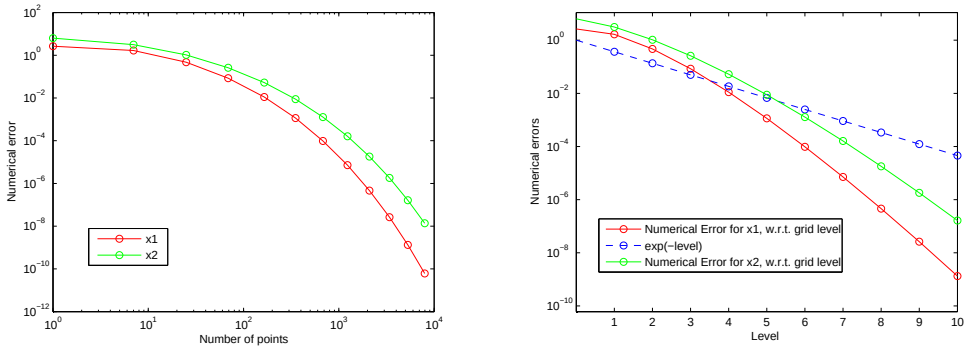
$$\text{IV}:\begin{cases} x_1'(t)=tx_1(t)+\beta x_1(t-1)+g_1(t),\\ x_2'(t)=\gamma x_1^2(t)+g_2(t),\\ x_1(t)=\gamma\exp^{-\alpha t},\quad -1\le t\le 0,\quad x_2(0)=1.\end{cases}$$

In problem III, α, β are two random variables. The functions $g_1(t,\alpha,\beta)$, $g_2(t,\alpha,\beta)$ and the initial conditions are chosen such that the exact solutions are $x_1=\sin(\alpha t)$, $x_2(t)=\cos(\beta t)$. We remark that the deterministic solver can be easily extended to deal with delay systems [1]. In problem IV, α,β,γ are traded as three random variables. This problem is a reduced version of the one used by Palanisamy et al. [17] for the optimal control of linear time-varying delay systems via single term Walsh series. Again, the functions $g_1(t)$, $g_2(t)$



(a) (b)

Figure 3: Example III, mean square error v.r.t. number of collocation points (a) and mean square error v.r.t sparse grid *level* (b).



(a) (b)

Figure 4: Example IV, mean square error v.r.t. number of collocation points (a) and mean square error v.r.t sparse grid *level* (b).

are chosen as

$$g_1(t) = -\left(\alpha\gamma\exp^{-\alpha t} + \gamma t\exp^{-\alpha t} + \beta\gamma\exp^{-\alpha(t-1)}\right), \quad g_2(t) = -2\beta\exp^{-2\beta t} - \gamma^3\exp^{-2\alpha t},$$

such that the exact solutions are $x_1(t) = \gamma\exp^{-\alpha t}$, $x_2(t) = \exp^{-2\beta t}$.

In Fig. 3, we provide with the numerical results for Example III, and again, the left plot is the mean square error with respect to the collocation points and the right plot is the mean square error with respect to the sparse grid *level*. We show the similar results for Example IV in Fig. 4. We learn from these plots that the proposed numerical methods works well for DDE systems. We also remark that for Example IV, where there are three stochastic variables, it involves a large number of collocation points (see Fig. 4, *Level=10* results in more or less 10^4 collocation points). One can lighten such an increase of number of points by adopting the so called anisotropic version of sparse grid [14]. We will investigate these techniques in the future study for large DDE systems.

5 Conclusions

In this work, we consider the numerical approach for delay differential equations with random coefficients. We show that the exact solutions of the problems considered can admit good regularity in the random space, provided that the given data satisfy some reasonable assumptions. A stochastic collocation method is proposed to approximate the solution in the the random space, and we use the Legendre spectral collocation method to solve the resulting deterministic delay differential equations. Convergence properties of the proposed method is analyzed. It is shown that the numerical method yields the familiar exponential order of convergence both in the random space and the time space. Numerical examples are given to illustrate these results.

We address the issue here and want to open up the possibility of designing efficient numerical methods to more complicate DDE systems, which include

- DDEs with discontinuous solutions in the time space and/or in the random space.
- DDE systems of large number of delay differential equations and large amount of random parameters.
- Investigate the stability properties of complicate non-linear delay problems with uncertainty input, such us DDEs in description of biochemical reactions channels [18], and Lotka-Volterra models for population Dynamics [19] and so on.

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