Journal of Computational Mathematics
Vol.xx, No.x, 2024, 1–23.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2402-m2023-0079

A DECOUPLED, LINEARLY IMPLICIT AND UNCONDITIONALLY ENERGY STABLE SCHEME FOR THE COUPLED CAHN-HILLIARD SYSTEMS*

Dan Zhao, Dongfang Li and Yanbin Tang[1)]
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China;
Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China
Email: dzhao@hust.edu.cn, dfli@mail.hust.edu.cn, tangyb@hust.edu.cn
Jinming Wen
College of Information Science and Technology, Jinan University, Guangzhou 510632, China
Email: jinming.wen@mail.mcgill.ca

Abstract

We present a decoupled, linearly implicit numerical scheme with energy stability and mass conservation for solving the coupled Cahn-Hilliard system. The time-discretization is done by leap-frog method with the scalar auxiliary variable (SAV) approach. It only needs to solve three linear equations at each time step, where each unknown variable can be solved independently. It is shown that the semi-discrete scheme has second-order accuracy in the temporal direction. Such convergence results are proved by a rigorous analysis of the boundedness of the numerical solution and the error estimates at different time-level. Numerical examples are presented to further confirm the validity of the methods.

Mathematics subject classification: 65M12, 65M22, 65M70.
Key words: Coupled Cahn-Hilliard system, Leap-frog method, Scalar auxiliary variable, Error estimate.

1. Introduction

In this paper, we consider the coupled Cahn-Hilliard system

$$\frac{\partial u}{\partial t} = M_u \Delta\big(- \epsilon_u^2 \Delta u + f(u,v)\big), \tag{1.1a}$$

$$\frac{\partial v}{\partial t} = M_v \big[\Delta\big(- \epsilon_v^2 \Delta v + g(u,v)\big) - \sigma(v - \bar{v})\big] \tag{1.1b}$$

in $\Omega \times (0,T]$ with periodic boundary conditions and following initial conditions:

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}), \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^d$ $(d = 1,2,3)$, M_u and M_v are the mobility constants that control the speed of u and v move. ϵ_u and ϵ_v represent the interfacial width between macrophases (described by u) and microphases (described by v). σ is related to the connectivity

* Received April 10, 2023 / Revised version received November 7, 2023 / Accepted February 19, 2024 / Published online May 6, 2024 /
1) Corresponding author

between the two components of the copolymer. $\bar{v} = \int_\Omega v d\mathbf{x}/|\Omega|$ is the mass ratio between two polymers. $f(u,v) = u^3 - u + \alpha v + \beta v^2$ and $g(u,v) = v^3 - v + \alpha u + 2\beta uv$ are the first variational derivative of the double welled potential

$$W(u,v) = \frac{1}{4}(u^2-1)^2 + \frac{1}{4}(v^2-1)^2 + \alpha uv + \beta uv^2,$$

where α and β are two coupling parameters. The coupled Cahn-Hilliard system was proposed to study the phase transition of the mixture of a homopolymer and a copolymer [2,3]. And it has been widely used in the study of physics and materials.

Theoretical analysis for the coupled Cahn-Hilliard system has been well done recently. Here we refer interested readers to [7,8,21] for the existence, uniqueness and regularity of solutions, weak formulation and the well-posedness of the system, and global existence and decay estimates to the system. It is noticed that system (1.1) has two distinguishing features. For one thing, taking the inner product of (1.1a) and (1.1b) with 1, we have

$$\frac{d}{dt}\int_\Omega u d\mathbf{x} = \frac{d}{dt}\int_\Omega v d\mathbf{x} = 0.$$

This indicates that the phase variable's mass are conserved. For another thing, taking the inner product of (1.1a) and (1.1b) with $-(-\Delta)^{-1}u_t$ and $-(-\Delta)^{-1}v_t$, respectively, we can obtain the energy dissipation law

$$\frac{d}{dt}E(u,v) = \int_\Omega \left[\frac{\delta E(u,v)}{\delta u}u_t + \frac{\delta E(u,v)}{\delta v}v_t\right] d\mathbf{x} = -\frac{1}{M_u}\|u_t\|_{-1}^2 - \frac{1}{M_v}\|v_t\|_{-1}^2 \leq 0 \tag{1.3}$$

with the energy defined by

$$E(u,v) = \int_\Omega \frac{\epsilon_u^2}{2}|\nabla u|^2 + \frac{\epsilon_v^2}{2}|\nabla v|^2 + W(u,v)d\mathbf{x} + \frac{\sigma}{2}\|(v-\bar{v})\|_{-1}^2, \tag{1.4}$$

where $\|\cdot\|_{-1}$ is the norm defined in H_{per}^{-1}.

In the past several decades, many efforts have been done to develop energy-stable schemes for the coupled Cahn-Hilliard system. Typical ways include the convex splitting approach [9,10], the stabilized approach [17,22,29], the Lagrange multiplier approach [4,5,12], the invariant energy quadratization approach [32,33,36], the scalar auxiliary variable (SAV) approach [1,11,14,18,25,26,28], the relaxation approach [13,15,16] and so on [6,23,27,34,35,37]. Generally speaking, a class of couple system was obtained by using the usual time-discretization with the mentioned approaches. For example, Yang and Kim [31] proposed a coupled scheme by using the Lagrange multiplier approach and the second-order backward difference formula (BDF2) method. Li *et al.* [24] presented a linearized implicit and coupled scheme by using the Crank-Nicolson-type method and a nonlinearly stabilized splitting approach. Li and Mei [20] developed a family of coupled schemes by using the BDF2 method and the SAV approach. The coupled schemes required some additional decoupled or iterative methods to get the numerical approximations.

In recent years, there are some decoupled schemes for solving the coupled Cahn-Hilliard system. In [19], a linearly implicit scheme was constructed by using an extension of the typical invariant energy quadratization approach with the Crank-Nicolson method, BDF1 method, and BDF2 method, respectively. In [30], a linearly implicit scheme was constructed by using BDF2 method and an efficient variant of the SAV approach and the energy relaxation technique. However, these references have no convergence results of the decoupled schemes.

This paper aims to present an effective structure-preserving scheme and its error estimates for the coupled Cahn-Hilliard system. Firstly, the coupled Cahn-Hilliard system is rewritten as an equivalent system by using the SAV approach. Secondly, the resulting system is approximated by using leap-frog method and stabilized approach for time discretization. A decoupled and semi-discrete scheme is obtained by carefully choosing the intermediate average variable in the leap-frog scheme. The semi-discrete scheme is proved to be unconditionally energy-stable and mass-conserving. Moreover, it is shown to be second-order accuracy in the temporal direction. Such convergence results are proved by rigorous error estimates and boundedness of the numerical solution at different time-level. Finally, several numerical examples are given to confirm the theoretical results.

The rest of the paper is organized as follows. In Section 2, we propose the semi-discrete scheme and prove the mass conservation and energy stability of the scheme. In Section 3, we give the error estimate of the semi-discrete scheme. In Section 4, some numerical examples are provided to validate the effectiveness of the scheme.

2. Numerical Scheme

In this section, we present the Leap-Frog-SAV scheme for solving system (1.1) and demonstrate that the scheme is unconditionally energy-stable and mass-conserving.

Firstly, we introduce some notations and definitions. Denote the space $L^p(\Omega)$ by L^p in short. The Sobolev spaces H^s will also be used. The space $L^p(0,T;V)$ represents the L^p space on the interval $(0,T)$ with values in the function space V. We use $\|\cdot\|_V$ to denote the norm in the space V, and the L^2 norm without a subscript. For any two functions $u,\ v\in L^2(\Omega)$, we denote the $L^2(\Omega)$ inner production and norm by

$$\langle u,v\rangle=\int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x},\quad \|u\|^2=\langle u,u\rangle.$$

Define the spaces

$$\begin{aligned}
&L_0^2(\Omega)=\{v\in L^2(\Omega)|\langle v,1\rangle=0\},\\
&L_{per}^2(\Omega)=\{v\in L^2(\Omega)|v \text{ is periodic on } \partial\Omega\},\\
&H_{per}^s(\Omega)=\{v\in H^s(\Omega)|v \text{ is periodic on } \partial\Omega\}.
\end{aligned}$$

Denote by $H_{per}^{-s}(\Omega)$ the dual space of $H_{per}^s(\Omega)$. Suppose $f,g\in L_0^2(\Omega)$, we define

$$v_f:=(-\Delta)^{-1}f\in H^2(\Omega)\cap L_0^2(\Omega)$$

is the unique solution to the periodic boundary value problem $-\Delta v_f=f$ in Ω. And in this case, we denote the inner product and norm in H_{per}^{-1} by

$$\langle f,g\rangle_{-1}=\langle\nabla v_f,\nabla v_g\rangle,\quad \|f\|_{-1}=\sqrt{\langle f,f\rangle_{-1}}.$$

And via integration by parts, we have

$$\langle f,g\rangle_{-1}=\langle(-\Delta)^{-1}f,g\rangle=\langle(-\Delta)^{-1}g,f\rangle=\langle g,f\rangle_{-1}.$$

Denote the unit operator by $\mathcal{I}$, which means $\mathcal{I}f=f$.

2.1. Leap-Frog-SAV scheme

Introduce the scalar auxiliary variable $r(t) = \sqrt{E_1(u,v) + C_0}$, where $C_0 > 0$ is a constant such that $E_1(u,v) + C_0 > 0$ and

$$E_1(u,v) := \int_\Omega W(u,v) - \frac{1}{2}S_1u^2 - \frac{1}{2}S_2v^2 d\mathbf{x}.$$

Here S_1 and S_2 are stabilization parameters. The system (1.1) can be rewritten as follows:

$$\frac{\partial u}{\partial t} = M_u\Delta\Big(- \epsilon_u^2\Delta u + S_1u + \frac{r(t)}{\sqrt{E_1(u,v)+C_0}}\tilde{f}(u,v)\Big), \tag{2.1a}$$

$$\frac{\partial v}{\partial t} = M_v\Big[\Delta\Big(- \epsilon_v^2\Delta v + S_2v + \frac{r(t)}{\sqrt{E_1(u,v)+C_0}}\tilde{g}(u,v)\Big) - \sigma(v-\bar{v})\Big], \tag{2.1b}$$

$$r_t = \frac{1}{2\sqrt{E_1(u,v)+C_0}}\big[\langle \tilde{f}(u,v), u_t\rangle + \langle \tilde{g}(u,v), v_t\rangle\big], \tag{2.1c}$$

where $\tilde{f} = f(u,v) - S_1u$, $\tilde{g} = g(u,v) - S_2v$. Next, we use leap-frog method to construct the time-discrete scheme for system (2.1).

For a positive integer N, let $\tau = T/N$ be the temporal stepsize, and $t_n = n\tau$, $n = 0,1,2,\ldots,N$, be the temporal grid points in $[0,T]$. Define $Q_\tau = \{t_n | 0 \le n \le N\}$. Denote U^n, V^n, R^n be the semi-discrete numerical approximations for $u(\cdot,t_n), v(\cdot,t_n)$, and $r(t_n)$, respectively. Denote

$$\bar{V}^n = \frac{1}{|\Omega|}\int_\Omega V^n d\mathbf{x}.$$

Let

$$F(u,v) = \frac{\tilde{f}(u,v)}{\sqrt{E_1(u,v)+C_0}}, \quad G(u,v) = \frac{\tilde{g}(u,v)}{\sqrt{E_1(u,v)+C_0}}.$$

Then, we have the Leap-Frog-SAV scheme as follows:

$$\begin{cases} \dfrac{U^{n+1}-U^{n-1}}{2\tau} = M_u\Big[- \epsilon_u^2\Delta^2\dfrac{U^{n+1}+U^{n-1}}{2} + S_1\Delta\dfrac{U^{n+1}+U^{n-1}}{2} + R^n\Delta F(U^n,V^n)\Big], & (2.2a) \\ \dfrac{V^{n+1}-V^{n-1}}{2\tau} = M_v\Big[- \epsilon_v^2\Delta^2\dfrac{V^{n+1}+V^{n-1}}{2} + S_2\Delta\dfrac{V^{n+1}+V^{n-1}}{2} + R^n\Delta G(U^n,V^n) & \\ \qquad\qquad - \sigma\Big(\dfrac{V^{n+1}+V^{n-1}}{2} - \dfrac{\bar{V}^{n+1}+\bar{V}^{n-1}}{2}\Big)\Big], & (2.2b) \\ \dfrac{R^{n+1}-R^{n-1}}{2\tau} = \dfrac{1}{2}\Big[\Big\langle F(U^n,V^n), \dfrac{U^{n+1}-U^{n-1}}{2\tau}\Big\rangle + \Big\langle G(U^n,V^n), \dfrac{V^{n+1}-V^{n-1}}{2\tau}\Big\rangle\Big], & (2.2c) \end{cases}$$

where $n = 1,2,\ldots,N-1$, and U^1, V^1, R^1 is solved by

$$\begin{cases} U^1 = U^0 + \tau M_u\big[- \epsilon_u^2\Delta^2U^1 + S_1\Delta U^1 + R^0\Delta F(U^0,V^0)\big], & (2.3a) \\ V^1 = V^0 + \tau M_v\big[- \epsilon_v^2\Delta^2V^1 + S_2\Delta V^1 + R^0\Delta G(U^0,V^0) - \sigma(V^1 - \bar{V}^1)\big], & (2.3b) \\ R^1 = R^0 + \dfrac{1}{2}\big[\langle F(U^0,V^0), U^1 - U^0\rangle + \langle G(U^0,V^0), V^1 - V^0\rangle\big], & (2.3c) \end{cases}$$

where $U^0 = u(\cdot,0)$, $V^0 = v(\cdot,0)$, $r^0 = \sqrt{E_1(u(\cdot,0),v(\cdot,0)) + C_0}$.

In fact, the value of $\bar{V}^n$ is equal to $\bar{v}(\cdot,0)$ because of the mass conservation, which will be proved later. And the scheme (2.2) can be efficiently implemented as follows:

- Compute U^1, V^1 and R^1 by solving Eqs. (2.3).
- For $1 \leq n \leq N$, we obtain U^{n+1} and V^{n+1} from scheme (2.2a)-(2.2b) by solving equations of the form

$$P_u U^{n+1} = Q_u U^{n-1} + 2\tau M_u \Delta F(U^n, V^n) R^n,$$
$$P_v V^{n+1} = Q_v V^{n-1} + 2\tau M_v \Delta G(U^n, V^n) R^n + 2\tau M_v \sigma \bar{v}^0,$$

where

$$P_u = \big[\mathcal{I} + \tau M_u \big(\epsilon_u^2 \Delta^2 - S_1 \Delta\big)\big], \quad P_v = \big[\mathcal{I} + \tau M_v \big(\epsilon_v^2 \Delta^2 - S_2 \Delta + \sigma \mathcal{I}\big)\big],$$
$$Q_u = \big[\mathcal{I} - \tau M_u \big(\epsilon_u^2 \Delta^2 - S_1 \Delta\big)\big], \quad Q_v = \big[\mathcal{I} - \tau M_v \big(\epsilon_v^2 \Delta^2 - S_2 \Delta + \sigma \mathcal{I}\big)\big].$$

- With U^{n+1} and V^{n+1} known, compute R^{n+1} by solving Eq. (2.2c).

2.2. Mass conservation and energy stability

In this subsection, we will give the proof of the mass conservation and energy stability of the proposed scheme.

Theorem 2.1. *Suppose that U^n, R^n are the numerical solutions of scheme* (2.2). *Then, for $1 \leq n \leq N$, we have*

$$\int_\Omega U^n d\mathbf{x} = \int_\Omega U^0 d\mathbf{x}, \quad \int_\Omega V^n d\mathbf{x} = \int_\Omega V^0 d\mathbf{x}, \quad E_C^n \leq E_C^{n-1},$$

where

$$E_C^0 = \frac{\epsilon_u^2}{2}\|\nabla U^0\|^2 + \frac{\epsilon_v^2}{2}\|\nabla V^0\|^2 + \frac{S_1}{2}\|U^0\|^2 + \frac{S_2}{2}\|V^0\|^2 + \frac{\sigma}{2}\|V^0 - \bar{V}^0\|_{-1}^2 + |R^0|^2,$$
$$E_C^n = \frac{\epsilon_u^2}{4}\big(\|\nabla U^n\|^2 + \|\nabla U^{n-1}\|^2\big) + \frac{\epsilon_v^2}{4}\big(\|\nabla V^n\|^2 + \|\nabla V^{n-1}\|^2\big) + \frac{S_1}{4}\big(\|U^n\|^2 + \|U^{n-1}\|^2\big)$$
$$+ \frac{S_2}{4}\big(\|V^n\|^2 + \|V^{n-1}\|^2\big) + \frac{\sigma}{4}\big(\|V^n - \bar{V}^n\|_{-1}^2 + \|V^{n-1} - \bar{V}^{n-1}\|_{-1}^2\big) + R^n R^{n-1}.$$

Proof. Taking the inner product of (2.3a) and (2.3b) with 1, it holds that

$$\langle U^1, 1\rangle = \langle U^0, 1\rangle + \tau M_u \big\langle \Delta(-\epsilon_u^2 \Delta U^1 + S_1 U^1 + R^0 F(U^0, V^0)), 1\big\rangle = \langle U^0, 1\rangle,$$
$$\langle V^1, 1\rangle = \langle V^0, 1\rangle + \tau M_v \big\langle \Delta(-\epsilon_v^2 \Delta V^1 + S_2 V^1 + R^0 G(U^0, V^0)), 1\big\rangle - \tau M_v \sigma \langle V^1 - \bar{V}^1, 1\rangle = \langle V^0, 1\rangle,$$

where we use the boundary conditions and

$$\langle V^1 - \bar{V}^1, 1\rangle = \int_\Omega V^1 - \bar{V}^1 d\mathbf{x} = \int_\Omega V^1 d\mathbf{x} - |\Omega| \bar{V}^1 = 0.$$

Taking the inner products of (2.2a) and (2.2b) with 1, we have

$$\langle U^{n+1} - U^{n-1}, 1\rangle = \tau M_u \big\langle \Delta\big[\big(-\epsilon_u^2 \Delta + S_1 \mathcal{I}\big)(U^{n+1} + U^{n-1}) + 2R^n F(U^n, V^n)\big], 1\big\rangle = 0,$$
$$\langle V^{n+1} - V^{n-1}, 1\rangle = \tau M_v \big\langle \Delta\big[\big(-\epsilon_v^2 \Delta + S_2 \mathcal{I}\big)(V^{n+1} + V^{n-1}) + 2R^n G(U^n, V^n)\big], 1\big\rangle$$
$$- \tau M_v \sigma \big[\langle V^{n+1} - \bar{V}^{n+1}, 1\rangle + \langle V^{n-1} - \bar{V}^{n-1}, 1\rangle\big]$$
$$= -\tau M_v \sigma \left[\int_\Omega V^{n+1} d\mathbf{x} - |\Omega| \bar{V}^{n+1} + \int_\Omega V^{n-1} d\mathbf{x} - |\Omega| \bar{V}^{n-1}\right] = 0.$$

Then, we have the mass conservation of the scheme in the sense that

$$\int_\Omega U^{n+1}d\mathbf{x} = \int_\Omega U^n d\mathbf{x} = \cdots = \int_\Omega U^1 d\mathbf{x} = \int_\Omega U^0 d\mathbf{x},$$
$$\int_\Omega V^{n+1}d\mathbf{x} = \int_\Omega V^n d\mathbf{x} = \cdots = \int_\Omega V^1 d\mathbf{x} = \int_\Omega V^0 d\mathbf{x}.$$

Next, we consider the discrete energy dissipation law of the scheme. Taking the inner products of (2.3a) and (2.3b) with $-(-\Delta)^{-1}(U^1 - U^0)/M_u\tau$ and $-(-\Delta)^{-1}(V^1 - V^0)/M_v\tau$, respectively, and adding them together, we have

$$\begin{aligned}
&-\frac{1}{M_u\tau}\|U^1 - U^0\|_{-1}^2 - \frac{1}{M_v\tau}\|V^1 - V^0\|_{-1}^2 \\
&= \epsilon_u^2\langle \nabla U^1, \nabla(U^1 - U^0)\rangle + \epsilon_v^2\langle \nabla V^1, \nabla(V^1 - V^0)\rangle \\
&\quad + S_1\langle U^1, U^1 - U^0\rangle + S_2\langle V^1, V^1 - V^0\rangle \\
&\quad + R^0\langle F(U^0, V^0), U^1 - U^0\rangle + R^0\langle G(U^0, V^0), V^1 - V^0\rangle \\
&\quad + \sigma\langle (-\Delta)^{-1}(V^1 - \bar{V}^1), V^1 - V^0\rangle.
\end{aligned} \tag{2.4}$$

Noting that $\int_\Omega V^1 - \bar{V}^1 d\mathbf{x} = 0$, i.e. $V^1 - \bar{V}^1 \in L_0^2(\Omega)$, we have $\langle (-\Delta)^{-1}(V^1 - \bar{V}^1), 1\rangle = 0$. Then, it holds that

$$\begin{aligned}
&\big\langle (-\Delta)^{-1}(V^1 - \bar{V}^1), V^1 - V^0\big\rangle \\
&= \big\langle (-\Delta)^{-1}(V^1 - \bar{V}^1), V^1 - V^0\big\rangle + (\bar{V}^1 - \bar{V}^0)\big\langle (-\Delta)^{-1}(V^1 - \bar{V}^1), 1\big\rangle \\
&= \big\langle (-\Delta)^{-1}(V^1 - \bar{V}^1), (V^1 - \bar{V}^1) - (V^0 - \bar{V}^0)\big\rangle \\
&= \frac{1}{2}\big[\|V^1 - \bar{V}^1\|_{-1}^2 - \|V^0 - \bar{V}^0\|_{-1}^2 + \|(V^1 - \bar{V}^1) - (V^0 - \bar{V}^0)\|_{-1}^2\big].
\end{aligned}$$

Multiplying $2R^0$ on both sides of (2.3c), we have

$$2\big(R^1R^0 - |R^0|^2\big) = R^0\langle F(U^0, V^0), U^1 - U^0\rangle + R^0\langle G(U^0, V^0), V^1 - V^0\rangle.$$

Taking them into (2.4), we obtain

$$\begin{aligned}
2(E_C^1 - E_C^0) &= \frac{\epsilon_u^2}{2}\big(\|\nabla U^1\|^2 - \|\nabla U^0\|^2\big) + \frac{\epsilon_v^2}{2}\big(\|\nabla V^1\|^2 - \|\nabla V^0\|^2\big) + \frac{S_1}{2}\big(\|U^1\|^2 - \|U^0\|^2\big) \\
&\quad + \frac{S_2}{2}\big(\|V^1\|^2 - \|V^0\|^2\big) + \frac{\sigma}{2}\big(\|V^1 - \bar{V}^1\|_{-1}^2 - \|V^0 - \bar{V}^0\|_{-1}^2\big) + R^1R^0 - |R^0|^2 \\
&\le -\frac{1}{M_u\tau}\|U^1 - U^0\|_{-1}^2 - \frac{1}{M_v\tau}\|V^1 - V^0\|_{-1}^2 \le 0.
\end{aligned}$$

Similarly, for $n \ge 1$, taking the inner products of (2.2a) and (2.2b) with $-(-\Delta)^{-1}(U^{n+1} - U^{n-1})/M_u$ and $-(-\Delta)^{-1}(V^{n+1} - V^{n-1})/M_v$, respectively, and adding them together, we obtain

$$\begin{aligned}
&-\frac{1}{2M_u\tau}\|U^{n+1} - U^{n-1}\|_{-1}^2 - \frac{1}{2M_v\tau}\|V^{n+1} - V^{n-1}\|_{-1}^2 \\
&= \frac{\epsilon_u^2}{2}\big(\|\nabla U^{n+1}\|^2 - \|\nabla U^{n-1}\|^2\big) + \frac{\epsilon_v^2}{2}\big(\|\nabla V^{n+1}\|^2 - \|\nabla V^{n-1}\|^2\big) \\
&\quad + \frac{S_1}{2}\big(\|U^{n+1}\|^2 - \|U^{n-1}\|^2\big) + \frac{S_2}{2}\big(\|V^{n+1}\|^2 - \|V^{n-1}\|^2\big) \\
&\quad + R^n\langle F(U^n, V^n), U^{n+1} - U^{n-1}\rangle + R^n\langle G(U^n, V^n), V^{n+1} - V^{n-1}\rangle \\
&\quad + \frac{\sigma}{2}\|V^{n+1} - \bar{V}^{n+1}\|_{-1}^2 - \|V^{n-1} - \bar{V}^{n-1}\|_{-1}^2,
\end{aligned}$$

where we use

$$\begin{aligned}&\big\langle(-\Delta)^{-1}\big[(V^{n+1}+V^{n-1})-(\bar V^{n+1}+\bar V^{n-1})\big],V^{n+1}-V^{n-1}\big\rangle\\=&\|V^{n+1}-\bar V^{n+1}\|_{-1}^2-\|V^{n-1}-\bar V^{n-1}\|_{-1}^2.\end{aligned}$$

Multiplying $4\tau R^n$ on both sides of (2.2c), we have

$$\begin{aligned}&2R^{n+1}R^n-2R^nR^{n-1}\\=&R^n\langle F(U^n,V^n),U^{n+1}-U^{n-1}\rangle+R^n\langle G(U^n,V^n),V^{n+1}-V^{n-1}\rangle.\end{aligned}$$

Then, we obtain

$$\begin{aligned}2\big(E_C^{n+1}-E_C^n\big)=&\frac{\epsilon_u^2}{2}\big(\|\nabla U^{n+1}\|^2-\|\nabla U^{n-1}\|^2\big)+\frac{\epsilon_v^2}{2}\big(\|\nabla V^{n+1}\|^2-\|\nabla V^{n-1}\|^2\big)\\&+\frac{S_1}{2}\big(\|U^{n+1}\|^2-\|U^{n-1}\|^2\big)+\frac{S_2}{2}\big(\|V^{n+1}\|^2-\|V^{n-1}\|^2\big)\\&+\frac{\sigma}{2}\big(\|V^{n+1}-\bar V^{n+1}\|_{-1}^2-\|V^{n-1}-\bar V^{n-1}\|_{-1}^2\big)+2R^{n+1}R^n-2R^nR^{n-1}\\=&-\frac{1}{2M_u\tau}\|U^{n+1}-U^{n-1}\|_{-1}^2-\frac{1}{2M_v\tau}\|V^{n+1}-V^{n-1}\|_{-1}^2\le 0,\end{aligned}$$

which implies that $E_C^{n+1}\le E_C^n$. This completes the proof. □

3. Error Estimate

In this section, we present the error estimate of the Leap-Frog-SAV scheme. Denote

$$u^n=u(\cdot,t_n),\quad v^n=v(\cdot,t_n),\quad \bar v^n=\frac{1}{|\Omega|}\int_\Omega v^n d\mathbf{x},\quad r^n=r(t_n).$$

Considering Eq. (2.1) at $t=t_n, n=1,2,\dots,N-1$, we have

$$\left\{\begin{aligned}&\frac{u^{n+1}-u^{n-1}}{2\tau}=M_u\Big[-\epsilon_u^2\Delta^2\frac{u^{n+1}+u^{n-1}}{2}+S_1\Delta\frac{u^{n+1}+u^{n-1}}{2}+r^n\Delta F(u^n,v^n)\Big]+\rho_1^n, &(3.1a)\\&\frac{v^{n+1}-v^{n-1}}{2\tau}=M_v\Big[-\epsilon_v^2\Delta^2\frac{v^{n+1}+v^{n-1}}{2}+S_2\Delta\frac{v^{n+1}+v^{n-1}}{2}+r^n\Delta G(u^n,v^n)\\&\qquad\qquad-\sigma\Big(\frac{v^{n+1}+v^{n-1}}{2}-\frac{\bar v^{n+1}+\bar v^{n-1}}{2}\Big)\Big]+\rho_2^n, &(3.1b)\\&\frac{r^{n+1}-r^{n-1}}{2\tau}=\frac12\Big[\Big\langle F(u^n,v^n),\frac{u^{n+1}-u^{n-1}}{2\tau}\Big\rangle+\Big\langle G(u^n,v^n),\frac{v^{n+1}-v^{n-1}}{2\tau}\Big\rangle\Big]+\rho_3^n. &(3.1c)\end{aligned}\right.$$

And for $t=0$, we have

$$\left\{\begin{aligned}&u^1=u^0+\tau M_u\big[-\epsilon_u^2\Delta^2u^1+S_1\Delta u^1+r^0\Delta F(u^0,u^0)\big]+\rho_1^0, &(3.2a)\\&v^1=v^0+\tau M_v\big[-\epsilon_v^2\Delta^2v^1+S_2\Delta v^1+r^0\Delta G(u^0,v^0)-\sigma(v^1-\bar v^1)\big]+\rho_2^0, &(3.2b)\\&r^1=r^0+\frac12\big[\langle F(u^0,v^0),u^1-u^0\rangle+\langle G(u^0,v^0),v^1-v^0\rangle\big]+\rho_3^0. &(3.2c)\end{aligned}\right.$$

The truncation errors ρ_i^n, $i=1,2,3$, are as follows:

$$\rho_1^0=u^1-u^0-\tau\partial_t u^0+\tau M_u\epsilon_u^2\Delta^2(u^1-u^0)-\tau M_uS_1\Delta(u^1-u^0)$$

$$
\begin{aligned}
&= \int_0^\tau (\tau - s) u_{tt}(\cdot, s) + \tau M_u \big(\epsilon_u^2 \Delta^2 u_t(\cdot, s) - S_1 \Delta u_t(\cdot, s)\big) ds, && (3.3)\\
\rho_2^0 &= v^1 - v^0 - \tau \partial_t v^0 + \tau M_v \epsilon_v^2 \Delta^2 (v^1 - v^0) - \tau M_v S_2 \Delta (v^1 - v^0) - \tau M_v \sigma (v^1 - v^0)\\
&= \int_0^\tau (\tau - s) v_{tt}(\cdot, s) + \tau M_v \big(\epsilon_v^2 \Delta^2 v_t(\cdot, s) - S_2 \Delta v_t(\cdot, s) - \sigma v_t(\cdot, s)\big) ds, && (3.4)\\
\rho_3^0 &= r^1 - r^0 - \tau \partial_t r^0 = \int_0^\tau (\tau - s) r_{tt}(s) ds, && (3.5)\\
\rho_1^n &= \frac{u^{n+1} - u^{n-1}}{2\tau} - \partial_t u^n - M_u \big(- \epsilon_u^2 \Delta^2 + S_1 \Delta \big) \left(\frac{u^{n+1} + u^{n-1}}{2} - u^n \right)\\
&= \int_{t_{n-1}}^{t_n} \frac{1}{4\tau} (s - t_{n-1})^2 u_{ttt}(\cdot, s) + \frac{M_u}{2} (s - t_{n-1}) \big(\epsilon_u^2 \Delta^2 u_{tt}(\cdot, s) - S_1 \Delta u_{tt}(\cdot, s)\big) ds\\
&\quad + \int_{t_n}^{t_{n+1}} \frac{1}{4\tau} (s - t_{n+1})^2 u_{ttt}(\cdot, s) - \frac{M_u}{2} (s - t_{n+1}) \big(\epsilon_u^2 \Delta^2 u_{tt}(\cdot, s) - S_1 \Delta u_{tt}(\cdot, s)\big) ds, && (3.6)\\
\rho_2^n &= \frac{v^{n+1} - v^{n-1}}{2\tau} - \partial_t v^n - M_v \big(- \epsilon_v^2 \Delta^2 + S_2 \Delta - \sigma \mathcal{I} \big) \left(\frac{v^{n+1} + v^{n-1}}{2} - v^n \right)\\
&\quad - M_v \sigma \left(\frac{\bar{v}^{n+1} + \bar{v}^{n-1}}{2} - \bar{v}^n \right)\\
&= \int_{t_{n-1}}^{t_n} \frac{1}{4\tau} (s - t_{n-1})^2 v_{ttt}(\cdot, s)\\
&\quad + \frac{M_v}{2} (s - t_{n-1}) \big((\epsilon_v^2 \Delta^2 - S_2 \Delta) v_{tt}(\cdot, s) + \sigma v_{tt}(\cdot, s) - \sigma \bar{v}_{tt}(\cdot, s) \big) ds\\
&\quad + \int_{t_n}^{t_{n+1}} \frac{1}{4\tau} (s - t_{n+1})^2 v_{ttt}(\cdot, s)\\
&\quad - \frac{M_v}{2} (s - t_{n+1}) \big((\epsilon_v^2 \Delta^2 - S_2 \Delta) v_{tt}(\cdot, s) + \sigma v_{tt}(\cdot, s) - \sigma \bar{v}_{tt}(\cdot, s) \big) ds, && (3.7)\\
\rho_3^n &= \frac{r^{n+1} - r^{n-1}}{2\tau} - \partial_t r^n\\
&\quad - \frac{1}{2} \left[\left\langle F(u^n, v^n), \frac{u^{n+1} - u^{n-1}}{2\tau} - \partial_t u^n \right\rangle + \left\langle G(u^n, v^n), \frac{v^{n+1} - v^{n-1}}{2\tau} - \partial_t v^n \right\rangle \right]\\
&= \frac{1}{4\tau} \left[\int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 r_{ttt}(s) ds + \int_{t_n}^{t_{n+1}} (s - t_{n+1})^2 r_{ttt}(s) ds \right]\\
&\quad - \frac{1}{8\tau} \left\langle F(u^n, v^n), \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 u_{ttt}(\cdot, s) ds + \int_{t_n}^{t_{n+1}} (s - t_{n+1})^2 u_{ttt}(\cdot, s) ds \right\rangle\\
&\quad - \frac{1}{8\tau} \left\langle G(u^n, v^n), \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 v_{ttt}(\cdot, s) ds + \int_{t_n}^{t_{n+1}} (s - t_{n+1})^2 v_{ttt}(\cdot, s) ds \right\rangle, && (3.8)
\end{aligned}
$$

where $\mathcal{I}$ refers to the identity operator, i.e. $\mathcal{I}v = v$, $v \in V$.

Denote that $e_u^n = u^n - U^n, e_v^n = v^n - V^n, e_r^n = r^n - R^n$. The following lemma is essential for the error estimate.

Lemma 3.1 (Discrete Grönwall Inequality). *Let τ, B and $a_k, b_k, c_k, \gamma_k\,(k > 0)$ be non-negative numbers such that*

$$
a_n + \tau \sum_{k=0}^{n} b_k \le \tau \sum_{k=0}^{n} \gamma_k a_k + \tau \sum_{k=0}^{n} c_k + B \quad \textit{for} \quad n \ge 0.
$$

Suppose that $\tau\gamma_k < 1$ *for all* k, *and set* $\sigma_k = (1-\tau\gamma_k)^{-1}$. *Then,*

$$a_n + \tau\sum_{k=0}^{n} b_k \le \left(\tau\sum_{k=0}^{n} c_k + B\right)\exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right).$$

Assumption 3.1. *Assume that exact solutions of the coupled Cahn-Hilliard system* (1.1)-(1.2) *exist and satisfy*

$$\begin{aligned}
&u, v \in L^\infty(0,T;W^{1,\infty}) \cap L^\infty(0,T;H^2), \quad && u_t, v_t \in L^\infty(0,T;L^2) \cap L^2(0,T;H^5),\\
&u_{tt} \in L^2(0,T;H^3), && v_{tt} \in L^2(0,T;H^3) \cap L^2(0,T;H^{-1}),\\
&u_{ttt}, v_{ttt} \in L^2(0,T;L^2) \cap L^2(0,T;H^{-1}).
\end{aligned}$$

And we denote that $M = \max\{\|u^n\|_{H^2}, \|\nabla u^n\|_{L^\infty}, \|v^n\|_{H^2}, \|\nabla v^n\|_{L^\infty}, \|u_t^n\|, \|v_t^n\|,\ |r^n|\}$.

Theorem 3.1. *Suppose that Assumption* 3.1 *holds, then, there exists a constant* τ_0 *such that, when* $\tau \le \tau_0$, *it holds for all* $0 \le n \le N$,

$$\max\big\{\|U^n\|_{H^2},\ \|V^n\|_{H^2}\big\} \le M + 1, \tag{3.9}$$

$$\max\big\{\|e_u^n\|_{H^1},\ \|e_v^n\|_{H^1},\ |e_r^n|\big\} \le C\tau^2, \tag{3.10}$$

where M *and* C *are constants independent on temporal step size* τ.

Proof. We complete the proof by applying the mathematical induction to n.

Step 1. The results (3.9)-(3.10) with $n = 0$ is valid since $e_u^0 = e_v^0 = e_r^0 = 0$. As for $n = 1$, subtracting (2.3) from (3.2), we have

$$\begin{cases}
e_u^1 = \tau M_u\big[-\epsilon_u^2\Delta^2 e_u^1 + S_1\Delta e_u^1\big] + \rho_1^0, & (3.11a)\\
e_v^1 = \tau M_v\big[-\epsilon_v^2\Delta^2 e_v^1 + S_2\Delta e_v^1 - \sigma e_v^1\big] + \rho_2^0, & (3.11b)\\
e_r^1 = \rho_3^0. & (3.11c)
\end{cases}$$

Taking the inner products with $e_u^1 - \Delta e_u^1$ and $e_v^1 - \Delta e_v^1$ of (3.11a) and (3.11b), respectively, we have

$$\begin{aligned}
&\big\|e_u^1\big\|_{H^1}^2 + \tau M_u S_1\big\|\nabla e_u^1\big\|^2 + \tau M_u\big(\epsilon_u^2 + S_1\big)\big\|\Delta e_u^1\big\|^2 + \tau M_u\epsilon_u^2\big\|\nabla\Delta e_u^1\big\|^2\\
=&\ \big\langle\rho_1^0, e_u^1\big\rangle + \big\langle\nabla\rho_1^0, \nabla e_u^1\big\rangle \le \frac{1}{2}\big\|\rho_1^0\big\|_{H^1}^2 + \frac{1}{2}\big\|e_u^1\big\|_{H^1}^2,\\
&(1+\tau M_v\sigma)\big\|e_v^1\big\|_{H^1}^2 + \tau M_v S_2\big\|\nabla e_v^1\big\|^2 + \tau M_v\big(\epsilon_v^2 + S_2\big)\big\|\Delta e_v^1\big\|^2 + \tau M_v\epsilon_v^2\big\|\nabla\Delta e_v^1\big\|^2\\
=&\ \big\langle\rho_2^0, e_v^1\big\rangle + \big\langle\nabla\rho_2^0, \nabla e_v^1\big\rangle \le \frac{1}{2}\big\|\rho_2^0\big\|_{H^1}^2 + \frac{1}{2}\big\|e_v^1\big\|_{H^1}^2.
\end{aligned}$$

From (3.3)-(3.5) we know that there exists a constant C independent of τ such that

$$\begin{aligned}
&\big\|\rho_1^0\big\|_{H^1} \le \tau\int_0^\tau \|u_{tt}\|_{H^1} + M_u\big(\epsilon_u^2\big\|\Delta^2 u_t\big\|_{H^1} + S_1\|\Delta u_t\|_{H^1}\big)ds \le C\tau^2,\\
&\big\|\rho_2^0\big\|_{H^1} \le \tau\int_0^\tau \|v_{tt}\|_{H^1} + M_v\big(\epsilon_v^2\big\|\Delta^2 v_t\big\|_{H^1} + S_2\|\Delta v_t\|_{H^1} + \sigma\|v_t\|_{H^1}\big)ds \le C\tau^2,\\
&\big|\rho_3^0\big| \le \tau\int_0^\tau |r_{tt}|ds \le C\tau^2.
\end{aligned}$$

Then, we have

$$\|e_u^1\|_{H^1}^2 + 2\tau M_u\big(\epsilon_u^2 + S_1\big)\|\Delta e_u^1\|^2 \le C^2\tau^4, \tag{3.12}$$
$$\|e_v^1\|_{H^1}^2 + 2\tau M_v\big(\epsilon_v^2 + S_2\big)\|\Delta e_v^1\|^2 \le C^2\tau^4. \tag{3.13}$$

Combining with

$$\big|e_r^1\big| = \big|\rho_3^0\big| \le C\tau^2, \tag{3.14}$$

we have (3.10) holds for $n = 1$. And

$$\|U^1\|_{H^2} \le \|u^1\|_{H^2} + \|e_u^1\|_{H^1} + \|\Delta e_u^1\| \le M + C\tau^2 + \frac{C}{\sqrt{2M_u(\epsilon_u^2 + S_1)}}\tau^{\frac{3}{2}} \le M + 1,$$
$$\|V^1\|_{H^2} \le \|v^1\|_{H^2} + \|e_v^1\|_{H^1} + \|\Delta e_v^1\| \le M + C\tau^2 + \frac{C}{\sqrt{2M_v(\epsilon_v^2 + S_2)}}\tau^{\frac{3}{2}} \le M + 1,$$

where we set

$$\tau \le \tau_1 := \min\left\{\frac{1}{\sqrt{2C}}, \frac{\sqrt[3]{2M_u(\epsilon_u^2 + S_1)}}{(2C)^{2/3}}, \frac{\sqrt[3]{2M_v(\epsilon_v^2 + S_2)}}{(2C)^{2/3}}\right\}.$$

Then, (3.9) holds for $n = 1$.

Step 2. Assuming that (3.9)-(3.10) hold for any $n\le m\,(1\le m\le N)$, and noting that $H^2 \hookrightarrow L^\infty$ in $\Omega \subseteq \mathbb{R}^d\,(d \le 3)$, we deduce that

$$\|u^n\|_{L^\infty}, \|U^n\|_{L^\infty}, \|v^n\|_{L^\infty}, \|V^n\|_{L^\infty} \le C_\Omega(M + 1), \quad 0 \le n \le m, \tag{3.15}$$

where C_Ω is a constant independent on τ. Therefore, we have the boundedness of the related functions. Denote that

$$\begin{aligned} M_1 = \max\big\{&\|\tilde f(u^m, v^m)\|, \|\tilde g(u^m, v^m)\|, \|\tilde f(U^m, V^m)\|, \|\tilde g(U^m, V^m)\|,\\ &\|\nabla\tilde f(u^m, v^m)\|, \|\nabla\tilde g(u^m, v^m)\|, \|\nabla\tilde f(U^m, V^m)\|, \|\nabla\tilde g(U^m, V^m)\|,\\ &\|F(U^m, V^m)\|, \|G(U^m, V^m)\|, \|\nabla F(U^m, V^m)\|, \|\nabla G(U^m, V^m)\|\big\}. \end{aligned}$$

Then, we have

$$\begin{aligned} &W(u^m, v^m) - W(U^m, V^m)\\ =& \frac{1}{4}\big[(u^m)^2 - 1\big] - \frac{1}{4}\big[(U^m)^2 - 1\big] + \frac{1}{4}\big[(v^m)^2 - 1\big] - \frac{1}{4}\big[(V^m)^2 - 1\big]\\ &+ \alpha u^m v^m - \alpha U^m V^m + \beta u^m (v^m)^2 - \beta U^m (V^m)^2\\ =& \frac{1}{4}\big[(u^m)^2 + (U^m)^2 - 2\big](u^m + U^m)e_u^m + \alpha v^m e_u^m + \alpha U^m e_v^m + \beta (v^m)^2 e_u^m\\ &+ \frac{1}{4}\big[(v^m)^2 + (V^m)^2 - 2\big](v^m + V^m)e_v^m + \beta U^1 (v^m + V^m)e_v^m, \end{aligned}$$

which implies that

$$\begin{aligned} \big|E_1(u^m, v^m) - E_1(U^m, V^m)\big| &\le \int_\Omega |W(u^m, v^m) - W(U^m, V^m)| d\mathbf{x}\\ &\le C_1\big(\|e_u^m\| + \|e_v^m\|\big), \end{aligned} \tag{3.16}$$

where C_1 is a constant dependent on α, β, M, and Ω. Then, we can obtain

$$\begin{aligned}
&\left|\frac{1}{\sqrt{E_1(u^m,v^m)+C_0}}-\frac{1}{\sqrt{E_1(U^m,V^m)+C_0}}\right| \\
=&\left|\frac{\sqrt{E_1(U^m,V^m)+C_0}-\sqrt{E_1(u^m,v^m)+C_0}}{\sqrt{\big(E_1(u^m,v^m)+C_0\big)\big(E_1(U^m,V^m)+C_0\big)}}\right| \\
\le&\frac{1}{C_0}\left|\frac{E_1(U^m,V^m)-E_1(u^m,v^m)}{\sqrt{E_1(u^m,v^m)+C_0}+\sqrt{E_1(U^m,V^m)+C_0}}\right| \\
\le&\frac{1}{2C_0\sqrt{C_0}}\big|E_1(u^m,v^m)-E_1(U^m,V^m)\big| \\
\le&\frac{C_1}{2C_0\sqrt{C_0}}\big(\|e_u^m\|+\|e_v^m\|\big), \\
&\big\|\tilde f(u^m,v^m)-\tilde f(U^m,V^m)\big\| \\
\le&\big\|\big[(u^m)^2+u^mU^m+(U^m)^2-1-S_1\big]e_u^m\big\|+\big\|\alpha e_v^m+\beta(v^m+V^m)e_v^m\big\| \\
\le&C_2\big(\|e_u^m\|+\|e_v^m\|\big), \\
&\big\|\nabla\tilde f(u^m,v^m)-\nabla\tilde f(U^m,V^m)\big\| \\
\le&\left\|\nabla u^m\left(\frac{\partial\tilde f}{\partial u}\Big|_{(u^m,v^m)}-\frac{\partial\tilde f}{\partial u}\Big|_{(U^m,V^m)}\right)\right\|+\left\|\nabla e_u^m\frac{\partial\tilde f}{\partial u}\Big|_{(U^m,V^m)}\right\| \\
&+\left\|\nabla v^m\left(\frac{\partial\tilde f}{\partial v}\Big|_{(u^m,v^m)}-\frac{\partial\tilde f}{\partial v}\Big|_{(U^m,V^m)}\right)\right\|+\left\|\nabla e_v^m\frac{\partial\tilde f}{\partial v}\Big|_{(U^m,V^m)}\right\| \\
\le&\|\nabla u^m\|_{L^\infty}\big\|3(u^m+U^m)e_u^m\big\|+\big\|\big[3(U^m)^2-1-S_1\big]\nabla e_u^m\big\| \\
&+\|\nabla v^m\|_{L^\infty}\big\|2\beta e_v^m\big\|+\big\|[\alpha+2\beta V^m]\nabla e_u^m\big\| \\
\le&C_2\big(\|e_u^m\|_{H^1}+\|e_v^m\|_{H^1}\big),
\end{aligned}$$

where $C_2=\max\{6C_\Omega M(M+1), 3C_\Omega^2(M+1)^2+1+S_1, \alpha+2\beta C_\Omega(M+1)\}$. Now, we have

$$\begin{aligned}
&\|F(u^m,v^m)-F(U^m,V^m)\| \\
=&\left\|\frac{\tilde f(u^m,v^m)}{\sqrt{E_1(u^m,v^m)+C_0}}-\frac{\tilde f(U^m,V^m)}{\sqrt{E_1(U^m,V^m)+C_0}}\right\| \\
\le&\left\|\tilde f(u^m,v^m)\left[\frac{1}{\sqrt{E_1(u^m,v^m)+C_0}}-\frac{1}{\sqrt{E_1(U^m,V^m)+C_0}}\right]\right\| \\
&+\left\|\frac{1}{\sqrt{E_1(U^m,V^m)+C_0}}\big[\tilde f(u^m,v^m)-\tilde f(U^m,V^m)\big]\right\| \\
\le&\left(\frac{C_1M_1}{2C_0\sqrt{C_0}}+\frac{C_2}{\sqrt{C_0}}\right)\big(\|e_u^m\|+\|e_v^m\|\big) \\
=:&\,M_{21}\big(\|e_u^m\|+\|e_v^m\|\big), \\
&\|\nabla F(u^m,v^m)-\nabla F(U^m,V^m)\| \\
=&\left\|\frac{\nabla\tilde f(u^m,v^m)}{\sqrt{E_1(u^m,v^m)+C_0}}-\frac{\nabla\tilde f(U^m,V^m)}{\sqrt{E_1(U^m,V^m)+C_0}}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \nabla \tilde{f}(u^m, v^m) \left[\frac{1}{\sqrt{E_1(u^m, v^m) + C_0}} - \frac{1}{\sqrt{E_1(U^m, V^m) + C_0}} \right] \right\| \\
&\quad + \left\| \left[\nabla \tilde{f}(u^m, v^m) - \nabla \tilde{f}(U^m, V^m) \right] \frac{1}{\sqrt{E_1(U^m, V^m) + C_0}} \right\| \\
&\leq M_{21} \left(\|e_u^m\|_{H^1} + \|e_v^m\|_{H^1} \right).
\end{aligned}
$$

Similarly, we can find a constant $M_{22} > 0$, such that

$$
\begin{aligned}
&\|G(u^m, v^m) - G(U^m, V^m)\| \leq M_{22} \left(\|e_u^m\| + \|e_v^m\| \right), \\
&\|\nabla G(u^m, v^m) - \nabla G(U^m, V^m)\| \leq M_{22} \left(\|e_u^m\|_{H^1} + \|e_v^m\|_{H^1} \right).
\end{aligned}
$$

Let $M_2 = 2(M_{21} + M_{22})^2$, we have

$$
\begin{aligned}
&\quad \|F(u^m, v^m) - F(U^m, V^m)\|^2 + \|G(u^m, v^m) - G(U^m, V^m)\|^2 \\
&\leq M_2 \Big(\|e_u^m\|^2 + \|e_v^m\|^2 \Big), \qquad (3.17) \\
&\quad \left\| \nabla \big(F(u^m, v^m) - F(U^m, V^m) \big) \right\|^2 + \left\| \nabla \big(G(u^m, v^m) - G(U^m, V^m) \big) \right\|^2 \\
&\leq M_2 \Big(\|e_u^m\|_{H^1}^2 + \|e_v^m\|_{H^1}^2 \Big). \qquad (3.18)
\end{aligned}
$$

Subtracting (2.2) from (3.1) with $n = m$, we have

$$
\begin{cases}
e_u^{m+1} - e_u^{m-1} = -\tau M_u \epsilon_u^2 \Delta^2 \big(e_u^{m+1} + e_u^{m-1} \big) + \tau M_u S_1 \Delta \big(e_u^{m+1} + e_u^{m-1} \big) \\
\qquad + 2\tau M_u \big[r^m \Delta F(u^m, v^m) - R^m \Delta F(U^m, V^m) \big] + 2\tau \rho_1^m, \qquad (3.19a) \\
e_v^{m+1} - e_v^{m-1} = -\tau M_v \epsilon_v^2 \Delta^2 \big(e_v^{m+1} + e_v^{m-1} \big) + \tau M_v S_2 \Delta \big(e_v^{m+1} + e_v^{m-1} \big) \\
\qquad - \tau M_v \sigma \big(e_v^{m+1} + e_v^{m-1} \big) \\
\qquad + 2\tau M_v \big[r^m \Delta G(u^m, v^m) - R^m \Delta G(U^m, V^m) \big] + 2\tau \rho_2^m, \qquad (3.19b) \\
e_r^{m+1} - e_r^{m-1} = \frac{1}{2} \big[\langle F(u^m, v^m), u^{m+1} - u^{m-1} \rangle - \langle F(U^m, V^m), U^{m+1} - U^{m-1} \rangle \big] \\
\qquad + \frac{1}{2} \big[\langle G(u^m, v^m), v^{m+1} - v^{m-1} \rangle - \langle G(U^m, V^m), V^{m+1} - V^{m-1} \rangle \big] + 2\tau \rho_3^m, \qquad (3.19c)
\end{cases}
$$

where ρ_i^m, $i = 1, 2, 3$, are defined in (3.6)-(3.8). Taking the inner products of (3.19a) and (3.19b) with $(e_u^{m+1} + e_u^{m-1}) - \Delta(e_u^{m+1} + e_u^{m-1})$ and $(e_v^{m+1} + e_v^{m-1}) - \Delta(e_v^{m+1} + e_v^{m-1})$, respectively, multiplying $2e_r^{m+1}$ on both sides of (3.19c), and adding up them together, we obtain

$$
\begin{aligned}
&\|e_u^{m+1}\|_{H^1}^2 - \|e_u^{m-1}\|_{H^1}^2 + \|e_v^{m+1}\|_{H^1}^2 - \|e_v^{m-1}\|_{H^1}^2 + |e_r^{m+1}|^2 - |e_r^{m-1}|^2 + |e_r^{m+1} - e_r^{m-1}|^2 \\
&\quad + \tau M_v \sigma \|e_v^{m+1} + e_v^{m-1}\|^2 + \tau M_u S_1 \|\nabla \big(e_u^{m+1} + e_u^{m-1} \big)\|^2 + \tau M_v (S_2 + \sigma) \|\nabla \big(e_v^{m+1} + e_v^{m-1} \big)\|^2 \\
&\quad + \tau M_u \big(S_1 + \epsilon_u^2 \big) \|\Delta \big(e_u^{m+1} + e_u^{m-1} \big)\|^2 + \tau M_v \big(S_2 + \epsilon_v^2 \big) \|\Delta \big(e_v^{m+1} + e_v^{m-1} \big)\|^2 \\
&\quad + \tau M_u \epsilon_u^2 \|\nabla \Delta \big(e_u^{m+1} + e_u^{m-1} \big)\|^2 + \tau M_v \epsilon_v^2 \|\nabla \Delta \big(e_v^{m+1} + e_v^{m-1} \big)\|^2 \\
&= 2\tau M_u \big\langle r^m \Delta F(u^m, v^m) - R^m \Delta F(U^m, V^m), \big(e_u^{m+1} + e_u^{m-1} \big) - \Delta \big(e_u^{m+1} + e_u^{m-1} \big) \big\rangle \\
&\quad + 2\tau M_v \big\langle r^m \Delta G(u^m, v^m) - R^m \Delta G(U^m, V^m), \big(e_v^{m+1} + e_v^{m-1} \big) - \Delta \big(e_v^{m+1} + e_v^{m-1} \big) \big\rangle \\
&\quad + e_r^{m+1} \big[\langle F(u^m, v^m), u^{m+1} - u^{m-1} \rangle - \langle F(U^m, V^m), U^{m+1} - U^{m-1} \rangle \big] \\
&\quad + e_r^{m+1} \big[\langle G(u^m, v^m), v^{m+1} - v^{m-1} \rangle - \langle G(U^m, V^m), V^{m+1} - V^{m-1} \rangle \big]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\tau\big\langle \rho_1^m, \big(e_u^{m+1}+e_u^{m-1}\big) - \Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&+ 2\tau\big\langle \rho_2^m, \big(e_v^{m+1}+e_v^{m-1}\big) - \Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\rangle + 4\tau\rho_3^m e_r^{m+1} \\
=:&\ A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7. \qquad (3.20)
\end{aligned}
$$

Noting that we have estimates in (3.15)-(3.18), and using Cauchy inequality and the inequality $ab \le \theta a^2 + b^2/(4\theta)$, each term at the right hand of (3.20) can be estimated step by step.

$$
\begin{aligned}
A_1 =&\ 2\tau M_u \big\langle r^m \Delta\big[F(u^m,v^m) - F(U^m,V^m)\big], \big(e_u^{m+1}+e_u^{m-1}\big) - \Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&+ 2\tau M_u \big\langle e_r^m \Delta F(U^m,V^m), \big(e_u^{m+1}+e_u^{m-1}\big) - \Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
=&\ -2\tau M_u \big\langle r^m \nabla\big[F(u^m,v^m) - F(U^m,V^m)\big], \nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&+ 2\tau M_u \big\langle r^m \nabla\big[F(u^m,v^m) - F(U^m,V^m)\big], \nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&- 2\tau M_u \big\langle e_r^m \nabla F(U^m,V^m), \nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&+ 2\tau M_u \big\langle e_r^m \nabla F(U^m,V^m), \nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
\le&\ 2\tau M_u \left(\frac{1}{2\theta_{11}}|r^m|^2\|\nabla F(u^m,v^m) - \nabla F(U^m,V^m)\|^2 + \frac{\theta_{11}}{2}\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2\right) \\
&+ 2\tau M_u \left(\frac{1}{2\theta_{21}}|r^m|^2\|\nabla F(u^m,v^m) - \nabla F(U^m,V^m)\|^2 + \frac{\theta_{21}}{2}\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2\right) \\
&+ 2\tau M_u \left(\frac{1}{2\theta_{12}}\|\nabla F(U^m,V^m)\|^2\big|e_r^m\big|^2 + \frac{\theta_{12}}{2}\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2\right) \\
&+ 2\tau M_u \left(\frac{1}{2\theta_{22}}\|\nabla F(U^m,V^m)\|^2\big|e_r^m\big|^2 + \frac{\theta_{22}}{2}\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2\right) \\
\le&\ \tau(\theta_{21}+\theta_{22})M_u\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2 + \tau(\theta_{11}+\theta_{12})M_u\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2 \\
&+ \tau\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}}\right)M_u M^2 M_2\big(\big\|e_u^m\big\|_{H^1} + \big\|e_u^m\big\|_{H^1}\big) + \tau\left(\frac{1}{\theta_{12}} + \frac{1}{\theta_{22}}\right)M_u M_1^2\big|e_r^m\big|^2, \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
A_2 =&\ 2\tau M_v \big\langle r^m \Delta\big[G(u^m,v^m) - G(U^m,V^m)\big], \big(e_v^{m+1}+e_v^{m-1}\big) - \Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\rangle \\
&+ 2\tau M_v \big\langle e_r^m \Delta G(U^m,V^m), \big(e_v^{m+1}+e_v^{m-1}\big) - \Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\rangle \\
\le&\ \tau(\eta_{21}+\eta_{22})M_v\big\|\nabla\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2 + \tau(\eta_{11}+\eta_{12})M_v\big\|\nabla\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2 \\
&+ \tau\left(\frac{1}{\eta_{11}} + \frac{1}{\eta_{21}}\right)M_v M^2 M_2\Big(\big\|e_u^m\big\|_{H^1}^2 + \big\|e_v^m\big\|_{H^1}^2\Big) + \tau\left(\frac{1}{\eta_{12}} + \frac{1}{\eta_{22}}\right)M_v M_1^2\big|e_r^m\big|^2. \qquad (3.22)
\end{aligned}
$$

As for

$$
\begin{aligned}
A_3 =&\ e_r^{m+1}\langle F(u^m,v^m) - F(U^m,V^m), u^{m+1} - u^{m-1}\rangle \\
&+ e_r^{m+1}\big\langle F(U^m,V^m), e_u^{m+1} - e_u^{m-1}\big\rangle := A_{31} + A_{32}, \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
A_4 =&\ e_r^{m+1}\langle G(u^m,v^m) - G(U^m,V^m), v^{m+1} - v^{m-1}\rangle \\
&+ e_r^{m+1}\big\langle G(U^m,V^m), e_v^{m+1} - e_v^{m-1}\big\rangle := A_{41} + A_{42}, \qquad (3.24)
\end{aligned}
$$

we have

$$
\begin{aligned}
A_{31} \le&\ 2\tau\left(\frac{1}{2}\big\|u_t^m\big\|^2\big|e_r^{m+1}\big|^2 + \frac{1}{2}\|F(u^m,v^m) - F(U^m,V^m)\|^2\right) \\
\le&\ \tau M^2\big|e_r^{m+1}\big|^2 + \tau M_2\Big(\big\|e_u^m\big\|^2 + \big\|e_v^m\big\|^2\Big), \qquad (3.25)
\end{aligned}
$$

$$
A_{41} \le \tau M^2\big|e_r^{m+1}\big|^2 + \tau M_2\Big(\big\|e_u^m\big\|^2 + \big\|e_v^m\big\|^2\Big). \qquad (3.26)
$$

And using (3.19a) and (3.19b), we have

$$
\begin{aligned}
A_{32} &= \tau M_u e_r^{m+1}\big\langle F(U^m,V^m), -\epsilon_u^2\Delta^2\big(e_u^{m+1}+e_u^{m-1}\big)+S_1\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&\quad + 2\tau M_u e_r^{m+1}\langle F(U^m,V^m), r^m\Delta F(u^m,v^m)-R^m\Delta F(U^m,V^m)\rangle \\
&\quad + 2\tau e_r^{m+1}\langle F(U^m,V^m),\rho_1^m\rangle \\
&\le \tau M_u\big\langle e_r^{m+1}\nabla F(U^m,V^m), \epsilon_u^2\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)-S_1\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle \\
&\quad - 2\tau M_u\big\langle e_r^{m+1}\nabla F(U^m,V^m), r^m\big[\nabla F(u^m,v^m)-\nabla F(U^m,V^m)\big]\big\rangle \\
&\quad - 2\tau M_u\big\langle e_r^{m+1}\nabla F(U^m,V^m), e_r^m\nabla F(U^m,V^m)\big\rangle + 2\tau\big\langle e_r^{m+1}F(U^m,V^m),\rho_1^m\big\rangle \\
&\le \tau M_u\epsilon_u^2\left(\frac{1}{4\theta_{23}}\|\nabla F(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2+\theta_{23}\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2\right) \\
&\quad + \tau M_u S_1\left(\frac{1}{4\theta_{13}}\|\nabla F(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2+\theta_{13}\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2\right) \\
&\quad + 2\tau M_u\left(\frac12\|\nabla F(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2+\frac12|r^m|^2\|\nabla F(u^m,v^m)-\nabla F(U^m,V^m)\|^2\right) \\
&\quad + 2\tau M_u\left(\frac12\|\nabla F(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2+\frac12\|\nabla F(U^m,V^m)\|^2\big|e_r^m\big|^2\right) \\
&\quad + 2\tau\left(\frac12\|\nabla F(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2+\frac12\big\|\rho_1^m\big\|_{-1}^2\right) \\
&= \tau\left[\left(\frac{1}{4\theta_{13}}S_1+\frac{1}{4\theta_{23}}\epsilon_u^2+2\right)M_u+1\right]\|\nabla F(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2 \\
&\quad + \tau M_u\|\nabla F(U^m,V^m)\|^2\big|e_r^m\big|^2 \\
&\quad + \tau\theta_{13}M_uS_1\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2+\tau\theta_{23}M_u\epsilon_u^2\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2 \\
&\quad + \tau M_u|r^m|^2\|\nabla F(u^m,v^m)-\nabla F(U^m,V^m)\|^2+\tau\big\|\rho_1^m\big\|_{-1}^2 \\
&\le \tau\left[\left(\frac{1}{4\theta_{13}}S_1+\frac{1}{4\theta_{23}}\epsilon_u^2+2\right)M_u+1\right]M_1^2\big|e_r^{m+1}\big|^2+\tau M_uM_1^2\big|e_r^m\big|^2 \\
&\quad + \tau\theta_{13}M_uS_1\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2+\tau\theta_{23}M_u\epsilon_u^2\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2 \\
&\quad + \tau M_uM^2M_2\big(\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2\big)+\tau\big\|\rho_1^m\big\|_{-1}^2, \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
A_{42} &= \tau M_v e_r^{m+1}\big\langle G(U^m,V^m), -\epsilon_v^2\Delta^2\big(e_v^{m+1}+e_v^{m-1}\big)+S_2\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\rangle \\
&\quad + 2\tau M_v e_r^{m+1}\langle G(U^m,V^m), r^m\Delta G(u^m,v^m)-R^m\Delta G(U^m,V^m)\rangle \\
&\quad - \tau M_v\sigma e_r^{m+1}\big\langle G(U^m,V^m),\big(e_v^{m+1}+e_v^{m-1}\big)\big\rangle+2\tau e_r^{m+1}\big\langle G(U^m,V^m),\rho_2^m\big\rangle \\
&\le \tau\left[\left(\frac{1}{4\eta_{13}}S_2+\frac{1}{4\eta_{23}}\epsilon_v^2+2\right)M_v+1\right]\|\nabla G(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2 \\
&\quad + \tau M_v\|\nabla G(U^m,V^m)\|^2\big|e_r^m\big|^2 \\
&\quad + \tau\eta_{13}M_vS_2\big\|\nabla\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2+\tau\eta_{23}M_v\epsilon_v^2\big\|\nabla\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2 \\
&\quad + \tau M_v|r^m|^2\|\nabla G(u^m,v^m)-\nabla G(U^m,V^m)\|^2+\tau\big\|\rho_2^m\big\|_{-1}^2 \\
&\quad + \tau M_v\sigma\left(\frac12\|G(U^m,V^m)\|^2\big|e_r^{m+1}\big|^2+\frac12\big\|e_v^{m+1}+e_v^{m-1}\big\|^2\right) \\
&\le \tau\left[\left(\frac{1}{4\eta_{13}}S_2+\frac{1}{4\eta_{23}}\epsilon_v^2+2+\frac12\sigma\right)M_v+1\right]M_1^2\big|e_r^{m+1}\big|^2+\tau M_vM_1^2\big|e_r^m\big|^2
\end{aligned}
$$

$$+\frac{\tau}{2}M_v\sigma\big\|e_v^{m+1}+e_v^{m-1}\big\|^2$$
$$+\tau\eta_{13}M_vS_2\big\|\nabla\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2+\tau\eta_{23}M_v\epsilon_v^2\big\|\nabla\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2$$
$$+\tau M_vM^2M_2\big(\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2\big)+\tau\big\|\rho_2^m\big\|_{-1}^2. \tag{3.28}$$

Substituting (3.25)-(3.28) into (3.23)-(3.24), we obtain

$$A_3\le\tau\bigg\{\bigg[\bigg(\frac{1}{4\theta_{13}}S_1+\frac{1}{4\theta_{23}}\epsilon_u^2+2\bigg)M_u+1\bigg]M_1^2+M^2\bigg\}\big|e_r^{m+1}\big|^2+\tau M_uM_1^2\big|e_r^m\big|^2$$
$$+\tau\theta_{13}M_uS_1\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2+\tau\theta_{23}M_u\epsilon_u^2\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2$$
$$+\tau\big(M_uM^2+1\big)M_2\big(\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2\big)+\tau\big\|\rho_1^m\big\|_{-1}^2, \tag{3.29}$$

$$A_4\le\tau\bigg\{\bigg[\bigg(\frac{1}{4\eta_{13}}S_2+\frac{1}{4\eta_{23}}\epsilon_v^2+2+\frac{1}{2}\sigma\bigg)M_v+1\bigg]M_1^2+M^2\bigg\}\big|e_r^{m+1}\big|^2+\tau M_vM_1^2\big|e_r^m\big|^2$$
$$+\frac{\tau}{2}M_v\sigma\big\|e_v^{m+1}+e_v^{m-1}\big\|^2+\tau\eta_{13}M_vS_2\big\|\nabla\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2+\tau\big\|\rho_2^m\big\|_{-1}^2$$
$$+\tau\eta_{23}M_v\epsilon_v^2\big\|\nabla\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2+\tau\big(M_vM^2+1\big)M_2\Big(\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2\Big). \tag{3.30}$$

Besides, we have

$$A_5=2\tau\big\langle\rho_1^m,\big(e_u^{m+1}+e_u^{m-1}\big)-\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\rangle$$
$$\le\tau\bigg(\frac{1}{\theta_{14}}+\frac{1}{\theta_{24}}\bigg)\big\|\rho_1^m\big\|_{-1}^2+\theta_{14}\tau\big\|\nabla\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2+\theta_{24}\tau\big\|\nabla\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2, \tag{3.31}$$

$$A_6=2\tau\big\langle\rho_2^m,\big(e_v^{m+1}+e_v^{m-1}\big)-\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\rangle$$
$$\le\tau\bigg(\frac{1}{\eta_{14}}+\frac{1}{\eta_{24}}\bigg)\big\|\rho_2^m\big\|_{-1}^2+\eta_{14}\tau\big\|\nabla\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2+\eta_{24}\tau\big\|\nabla\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2, \tag{3.32}$$

$$A_7\le4\tau\bigg(\frac{1}{2}\big|\rho_3^m\big|^2+\frac{1}{2}\big|e_r^{m+1}\big|^2\bigg)=2\tau\big(\big|\rho_3^m\big|^2+\big|e_r^{m+1}\big|^2\big). \tag{3.33}$$

Then, we let parameters θ_{ij} and η_{ij}, $i=1,2$, $j=1,2,3,4$, satisfy

$$(\theta_{11}+\theta_{12}+\theta_{13}S_1)M_u+\theta_{14}=M_uS_1,\quad(\eta_{11}+\eta_{12}+\eta_{13}S_2)M_v+\eta_{14}=M_v(S_2+\sigma),$$
$$(\theta_{21}+\theta_{22}+\theta_{23}\epsilon_u^2)M_u+\theta_{24}=M_u\epsilon_u^2,\quad(\eta_{21}+\eta_{22}+\eta_{23}\epsilon_v^2)M_v+\eta_{24}=M_v\epsilon_v^2.$$

Substituting (3.21)-(3.33) into (3.20), we have

$$\big\|e_u^{m+1}\big\|_{H^1}^2-\big\|e_u^{m-1}\big\|_{H^1}^2+\big\|e_v^{m+1}\big\|_{H^1}^2-\big\|e_v^{m-1}\big\|_{H^1}^2+\big|e_r^{m+1}\big|^2-\big|e_r^{m-1}\big|^2$$
$$+\tau M_u\big(S_1+\epsilon_u^2\big)\big\|\Delta\big(e_u^{m+1}+e_u^{m-1}\big)\big\|^2+\tau M_v\big(S_2+\epsilon_v^2\big)\big\|\Delta\big(e_v^{m+1}+e_v^{m-1}\big)\big\|^2$$
$$\le\tau\bigg\{\bigg[\bigg(\frac{1}{4\theta_{13}}S_1+\frac{1}{4\theta_{23}}\epsilon_u^2+2\bigg)M_u$$
$$+\bigg(\frac{1}{4\eta_{13}}S_2+\frac{1}{4\eta_{23}}\epsilon_v^2+\frac{1}{2}\sigma+2\bigg)M_v+2\bigg]M_1^2+2(M^2+1)\bigg\}\big|e_r^{m+1}\big|^2$$
$$+\tau M_1^2\bigg[\bigg(\frac{1}{\theta_{12}}+\frac{1}{\theta_{22}}+1\bigg)M_u+\bigg(\frac{1}{\eta_{12}}+\frac{1}{\eta_{22}}+1\bigg)M_v\bigg]\big|e_r^m\big|^2+\tau\bigg(\frac{1}{\theta_{14}}+\frac{1}{\theta_{24}}+1\bigg)\big\|\rho_1^m\big\|_{-1}^2$$
$$+\tau M_2\bigg\{\bigg[\bigg(\frac{1}{\theta_{11}}+\frac{1}{\theta_{21}}+1\bigg)M_u+\bigg(\frac{1}{\eta_{11}}+\frac{1}{\eta_{21}}+1\bigg)M_v\bigg]M^2+2\bigg\}\Big(\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2\Big)$$

$$+\tau\Big(\frac{1}{\eta_{14}}+\frac{1}{\eta_{24}}+1\Big)\big\|\rho_2^m\big\|_{-1}^2+2\tau\big|\rho_3^m\big|^2$$
$$\leq C_3\tau\Big(\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2+\big|e_r^m\big|^2+\big|e_r^{m+1}\big|^2\Big)+C_4\tau\Big(\big\|\rho_1^m\big\|_{-1}^2+\big\|\rho_2^m\big\|_{-1}^2+\big|\rho_3^m\big|^2\Big),$$

where C_3 takes the maximum value of the coefficients in front of $|e_r^{m+1}|^2, |e_r^m|^2, \|e_u^m\|_{H^1}^2 + \|e_v^m\|_{H^1}^2$, and

$$C_4=\max\left\{\left(\frac{1}{\theta_{14}}+\frac{1}{\theta_{24}}+1\right),\left(\frac{1}{\eta_{14}}+\frac{1}{\eta_{24}}+1\right),2\right\}.$$

Let

$$G^m=\big\|e_u^{m+1}\big\|_{H^1}^2+\big\|e_u^m\big\|_{H^1}^2+\big\|e_v^{m+1}\big\|_{H^1}^2+\big\|e_v^m\big\|_{H^1}^2+\big|e_r^{m+1}\big|^2+\big|e_r^m\big|^2.$$

Replacing m with k, and summing k from 1 to m, we obtain

$$G^m+\tau\sum_{k=1}^m\big[M_u\big(S_1+\epsilon_u^2\big)\big\|\Delta\big(e_u^{k+1}+e_u^{k-1}\big)\big\|^2+M_v\big(S_2+\epsilon_v^2\big)\big\|\Delta\big(e_v^{k+1}+e_v^{k-1}\big)\big\|^2\big]$$
$$\leq G^0+C_3\tau\sum_{k=1}^m G^k+C_4\tau\sum_{k=1}^m\Big(\big\|\rho_1^k\big\|_{-1}^2+\big\|\rho_2^k\big\|_{-1}^2+\big|\rho_3^k\big|^2\Big). \tag{3.34}$$

According to (3.6)-(3.8), we know that there exists constants K_i, $i=1,2,3$, such that

$$\big\|\rho_1^n\big\|_{-1}^2\leq K_1\tau^3\Big(\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|_{-1}^2+\|u_{tt}\|_{H^3}^2+\|u_{tt}\|_{H^1}^2ds\Big),$$
$$\big\|\rho_2^n\big\|_{-1}^2\leq K_2\tau^3\Big(\int_{t_{n-1}}^{t_{n+1}}\|v_{ttt}\|_{-1}^2+\|v_{tt}\|_{H^3}^2+\|v_{tt}\|_{H^1}^2+\|v_{tt}\|_{-1}^2ds\Big),$$
$$\big|\rho_3^n\big|^2\leq K_3\tau^3\Big(\int_{t_{n-1}}^{t_{n+1}}|r_{ttt}|^2+\|u_{ttt}\|^2+\|v_{ttt}\|^2ds\Big).$$

Combining with Assumption 3.1, there exists a constant C_5 such that

$$\big\|\rho_1^n\big\|_{-1}^2+\big\|\rho_2^n\big\|_{-1}^2+\big|\rho_3^n\big|^2\leq C_5\tau^4.$$

Besides, from (3.12)-(3.14), we have

$$G^0=\big\|e_u^1\big\|_{H^1}^2+\big\|e_v^1\big\|_{H^1}^2+\big|e_r^1\big|^2\leq\left(\frac{C_1^2}{C_0}+2\right)M^2\tau^4.$$

Applying the discrete Grönwall inequality in Lemma 3.1 to (3.34). Then, for

$$\tau\leq\tau_2=\min\left\{\frac{1}{C_3},\tau_1\right\},$$

we have

$$G^m+\tau\sum_{k=1}^m\Big[M_u\big(S_1+\epsilon_u^2\big)\big\|\Delta\big(e_u^{k+1}+e_u^{k-1}\big)\big\|^2+M_v\big(S_2+\epsilon_v^2\big)\big\|\Delta\big(e_v^{k+2}+e_v^{k}\big)\big\|^2\Big]$$
$$\leq\exp\big\{C_3T(1-C_3\tau)^{-1}\big\}\left[C_4\tau\sum_{k=0}^m\big(\big\|(-\Delta)^{-\frac12}\rho_1^k\big\|^2+\big\|(-\Delta)^{-\frac12}\rho_2^k\big\|^2+\big|\rho_3^k\big|^2\big)+G^0\right]$$
$$\leq\exp\big\{C_3T(1-C_3\tau)^{-1}\big\}\left[C_4C_5T+\left(\frac{C_1^2}{C_0}+2\right)M^2\right]\tau^4\leq C^2\tau^4, \tag{3.35}$$

where

$$C^2 = \max\left\{ e^{\frac{C_3 T}{1-C_3}} \left[C_4 C_5 T + \left(\frac{C_1^2}{C_0} + 2 \right) M^2 \right],\ M^2,\ \frac{C_1^2 M^2}{C_0} \right\}.$$

It implies that

$$\left\| e_u^{m+1} \right\|_{H^1}^2 + \left\| e_v^{m+1} \right\|_{H^1}^2 + \left| e_r^{m+1} \right|^2 \le G^m \le C^2 \tau^4.$$

And the results (3.10) hold for $n = m+1$. Thus, by the mathematical induction, we have (3.10) holds for all $n \ge 0$.

Step 3. According to (3.35), we have

$$\sum_{k=1}^{m} \left[M_u \left(S_1 + \epsilon_u^2\right) \left\| \Delta\left(e_u^{k+1} + e_u^{k-1}\right) \right\|^2 + M_v \left(S_2 + \epsilon_v^2\right) \left\| \Delta\left(e_v^{k+1} + e_v^{k-1}\right) \right\|^2 \right] \le C^2 \tau^3.$$

It implies that

$$\left\| \Delta\left(e_u^{m+1} + e_u^{m-1}\right) \right\|, \left\| \Delta\left(e_v^{k+1} + e_v^{k-1}\right) \right\| \le C_6 \tau^{\frac{3}{2}},$$

where

$$C_6 = \max\left\{ \frac{C}{\sqrt{M_u (S_1 + \epsilon_u^2)}}, \frac{C}{\sqrt{M_v (S_2 + \epsilon_v^2)}} \right\}.$$

Then, we have

$$\begin{aligned} &\left\| \Delta e_u^{m+1} \right\| \le \left\| \Delta\left(e_u^{m+1} + e_u^{m-1}\right) \right\| + \left\| \Delta\left(e_u^{m-1} + e_u^{m-3}\right) \right\| + \cdots \le C_6 (m+1) \tau^{\frac{3}{2}} \le C_6 T \sqrt{\tau}, \\ &\| \Delta U^{m+1} \| \le \| \Delta u^{m+1} \| + \left\| \Delta e_u^{m+1} \right\| \le M + C_6 T \sqrt{\tau}. \end{aligned}$$

It follows that when $\tau \le \tau_0 = \min\{1/(4(C_6 T)^2), \tau_2\}$,

$$\| U^{m+1} \|_{H^2} \le \| u^{m+1} \|_{H^2} + \left\| e_u^{m+1} \right\|_{H^1} + \left\| \Delta e_u^{m+1} \right\| \le M + 1.$$

Similarly, we can obtain $\|V^{m+1}\|_{H^2} \le M + 1$, which implies that (3.9) holds for all $n \ge 0$ and we finish the proof. □

4. Numerical Examples

In this section, we present some numerical examples to test the convergence order, mass conservation and energy stability. The spatial discretization is done by Fourier spectral method. All computations are performed by using the software Matlab. In numerical experiments, the convergence orders are calculated by

$$order = \log_2 \left(\frac{err(2\tau)}{err(\tau)} \right),$$

where $err(\tau)$ is the numerical error computed with step size τ.

4.1. Convergence test

Consider the following problem in $\Omega \times [0, T]$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = M_u \Delta\big(-\epsilon_u^2 \Delta u + f(u,v) \big) + f_1, \\ \dfrac{\partial v}{\partial t} = M_v \big[\Delta\big(-\epsilon_v^2 \Delta v + g(u,v) \big) - \sigma(v - \bar{v}) \big] + g_1, \\ u(x,y,0) = v(x,y,0) = 0.25 \sin(x) \sin(y) + 0.1 \end{cases} \tag{4.1}$$

with periodic boundary conditions, where $\Omega = [-3\pi, \pi] \times [-3\pi, \pi]$, $T = 0.05, M_u = 1, M_v = 1$, $\alpha = 2, \beta = 3, \sigma = 50$, f_1 and g_1 are chosen such that the problem has an exact solution

$$u = v = 0.25\sin(x)\sin(y)\cos(t) + 0.1.$$

Here we use Fourier spectral method with 32×32 nodes in the spatial direction and Leap-Frog-SAV scheme in the temporal direction to solve the problem. The stability parameters in the scheme are set as $S_1 = S_2 = C_0 = 5$. To test the temporal convergence orders, the problem is solved with different time step sizes $\tau = T/N$, $N = 4, 8, 16, 32, 64, 128$. Besides, to test the stability of the scheme, the problem with $\epsilon_u^2 = \epsilon_v^2 = 0.1$ and $\epsilon_u^2 = \epsilon_v^2 = 0.01$ are both solved.

Table 4.1 shows numerical results in the case of $\epsilon_u^2 = \epsilon_v^2 = 0.1$, including the maximum numerical errors of u and v at time T, temporal convergence orders, and computational time. Table 4.2 shows numerical results in the case of $\epsilon_u^2 = \epsilon_v^2 = 0.01$. From the results, we can see that the problem in both cases can be solved well by the Leap-Frog-SAV scheme. And we can see from the tables that the convergence order in the temporal direction of the Leap-Frog-SAV scheme is of 2. It is the same as the analysis results in Section 3. Besides, the computational time required for calculation is very short, which implies that our decoupled scheme only needs a little computational cost. Fig. 4.1 shows the efficiency curves of maximum numerical errors at T versus computational time, where the convergence orders can be seen intuitively.

Table 4.1: Numerical errors, convergence orders in the temporal direction, and computational time for problem (4.1) with $\epsilon_u^2 = \epsilon_v^2 = 0.1$.

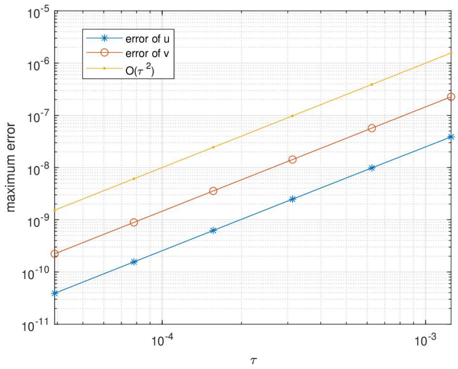
N	Error of u	Order of u	Error of v	Order of v	Computational time
4	3.87e-08	*	2.27e-07	*	5.12e-02
8	9.84e-09	1.97	5.68e-08	2.00	6.51e-02
16	2.48e-09	1.99	1.42e-08	2.00	1.21e-01
32	6.23e-10	1.99	3.56e-09	2.00	2.34e-01
64	1.56e-10	2.00	8.90e-10	2.00	4.97e-01
128	3.91e-11	2.00	2.23e-10	2.00	9.40e-01

Table 4.2: Numerical errors, convergence orders in the temporal direction, and computational time for problem (4.1) with $\epsilon_u^2 = \epsilon_v^2 = 0.01$.

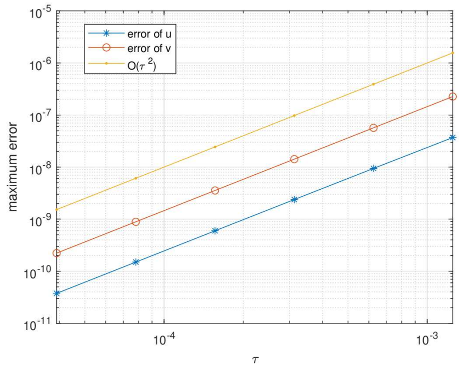
N	Error of u	Order of u	Error of v	Order of v	Computational time
4	3.69e-08	*	2.26e-07	*	5.35e-02
8	9.43e-09	1.97	5.68e-08	1.99	7.12e-01
16	2.38e-09	1.99	1.42e-08	2.00	1.27e-01
32	5.98e-10	1.99	3.56e-09	2.00	2.66e-01
64	1.50e-10	2.00	8.90e-10	2.00	5.31e-01
128	3.76e-11	2.00	2.22e-10	2.00	9.87e-01

4.2. Test on energy stability and mass conservation

Consider the original system (1.1)-(1.2) in $\Omega \times (0, T)$, where we set $\Omega = [0, 1] \times [0, 1]$, $\epsilon_u = \epsilon_v = 0.05, M_u = 1, M_v = 0.05, \alpha = 0.01, \beta = -0.9, \sigma = 100$. We investigated the energy



(a)



(b)

Fig. 4.1. The maximum numerical errors with $N = 4, 8, 16, 32, 64, 128$ for problem (4.1) with (a) $\epsilon_u^2 = \epsilon_v^2 = 0.1$ and (b) $\epsilon_u^2 = \epsilon_v^2 = 0.01$.

stability, mass conservation, and phase transition behaviors of the copolymer and homopolymer mixtures in this subsection with two smooth initial data. The problem is solved by Fourier spectral method with 64×64 nodes for the spatial discretization and Leap-Frog-SAV scheme with $S_1 = S_2 = C_0 = 10$ in the temporal direction. Here we define the error of mass as

$$\text{error of mass}(t) = |m_\phi(t) - m_\phi(0)|,$$

where $m_\phi(t)$ denotes the mass of ϕ at time t (with $\phi = u$ or v).

Firstly, we choose the initial functions as

$$u_0 = \sin\big(2x(x-1)y(y-1)\big), \quad v_0 = \cos\big(10(x-y)\big)x(x-1)y(y-1). \tag{4.2}$$

Fig. 4.2 shows the errors of mass of u and v with $\tau = 1e-4$. From the figure we can see that the errors of mass of u and v are both in a very small magnitude, indicating that the scheme is mass-conserving. Fig. 4.3 displays the discrete energy with $\tau =$ 1e-4, 5e-5, 1e-5, and 5e-6. It can be seen clearly from the figure that the scheme follows the discrete energy dissipation law. Furthermore, we display the discrete energy with $S_1 = S_2 = 0$ (without stability parameters), we can see that the numerical solutions blow up. These numerical results corroborate that the stabilization terms can improve the stability of the numerical scheme.

Besides, we present the phase evolution of phase variables u and v at $t = 0, 0.2, 0.3, 0.8, 50$ in Fig. 4.4. From the results, we can see that there is a macro-phase separation between homopolymer and copolymer (described by u), and a micro-phase separation of the diblock copolymer (described by v) occurs inside the separated domain. The results are the same as the numerical results in [20]. It verifies the effectiveness of our scheme.

Secondly, we choose another two initial functions as

$$u_0 = \sin\big(2x(x-1)y(y-1)\big), \quad v_0 = \cos\big(40(x-y)\big)x(x-1)y(y-1) \tag{4.3}$$

to verify the energy stability and mass conservation of the scheme. The discrete energy with $\tau =$ 1e-4, 5e-5, 1e-5, and 5e-6 are displayed in Fig. 4.6. And the errors of mass of u and v with $\tau =$ 1e-4 are shown in Fig. 4.5. Those results also imply that the Leap-Frog-SAV scheme is energy-stable and mass-conserving. In Fig. 4.7, we present more results about the phase evolution of phase variables u and v at $t = 0, 0.3, 0.8, 37, 50$. As can be seen from figures, the phase field separation occurs in a short time and gradually stabilizes. The results are the same as the numerical results in [20]. And they confirm the effectiveness of the scheme.

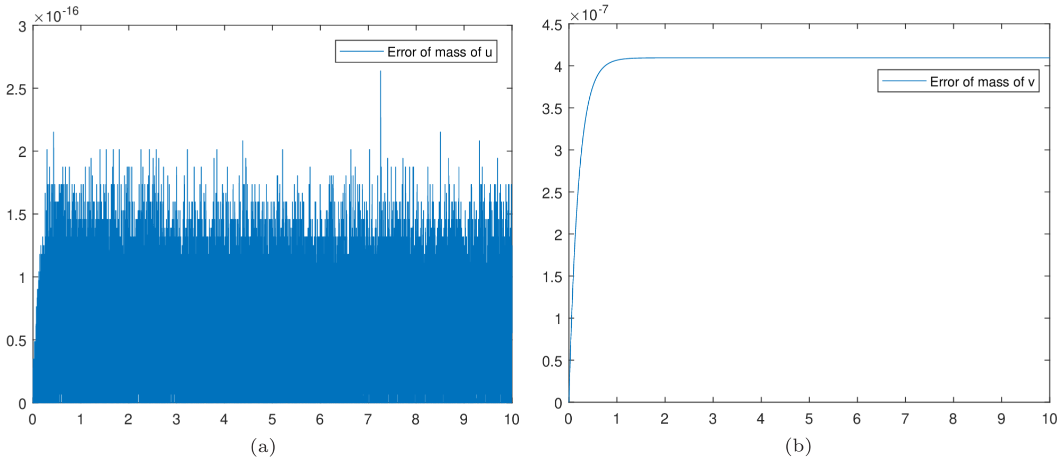


Fig. 4.2. The error of mass of u (a) and the error of mass of v (b) with τ = 1e-4 for the problem with initial data (4.2).

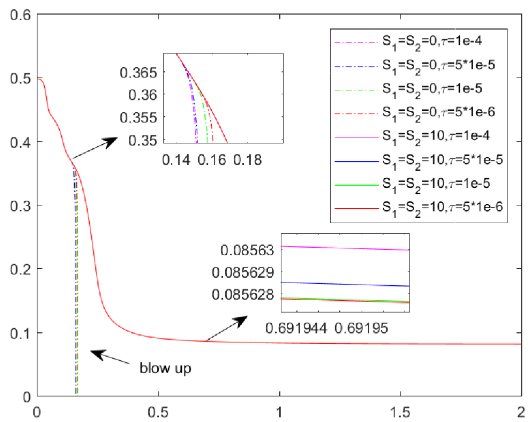


Fig. 4.3. The discrete energy with τ = 1e-4, 5e-5, 1e-5, 5e-6 for the problem with initial data (4.2).

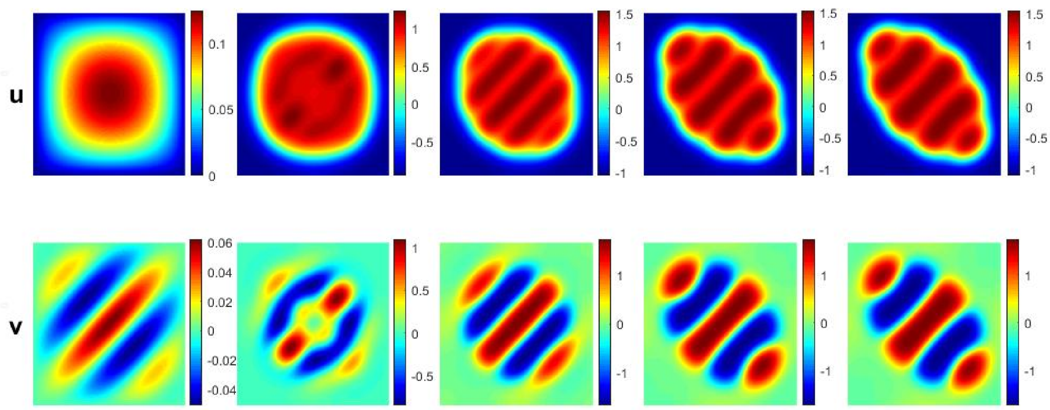


Fig. 4.4. Evolution of u (top) and v (bottom) with τ = 1e-04 for the problem with initial data (4.2). From left to right: $t = 0, 0.2, 0.3, 0.8, 50$.

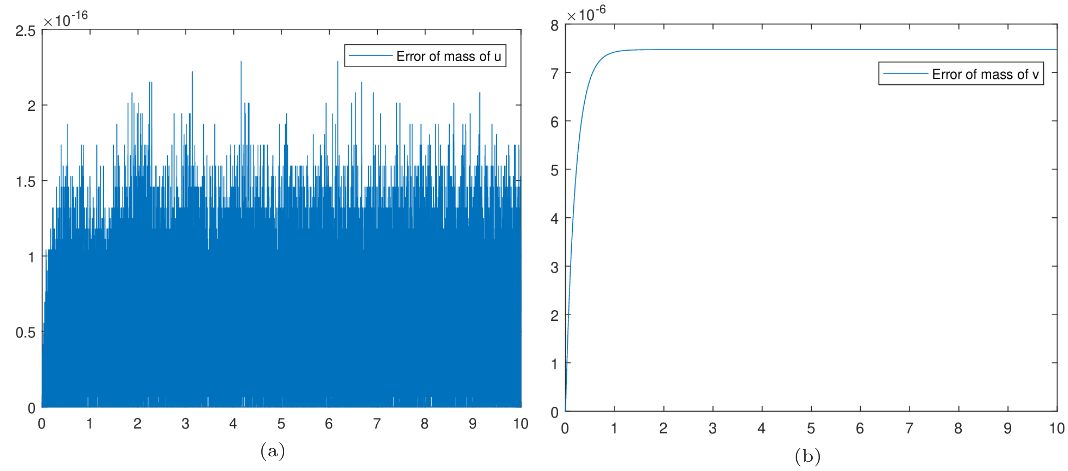


Fig. 4.5. The error of mass of u (a) and the error of mass of v (b) with τ = 1e-4 for the problem with initial data (4.3).

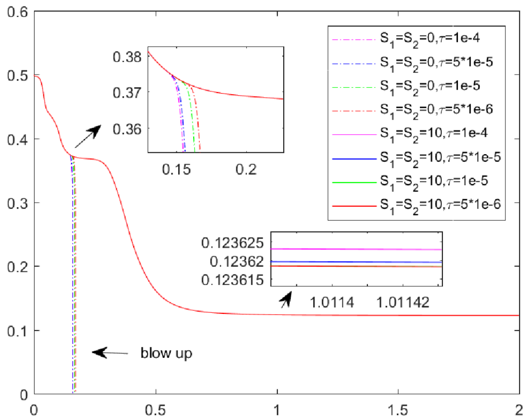


Fig. 4.6. The energy curves with τ = 1e-4, 5e-5, 1e-5, 5e-6 for solving the problem with initial data (4.3).

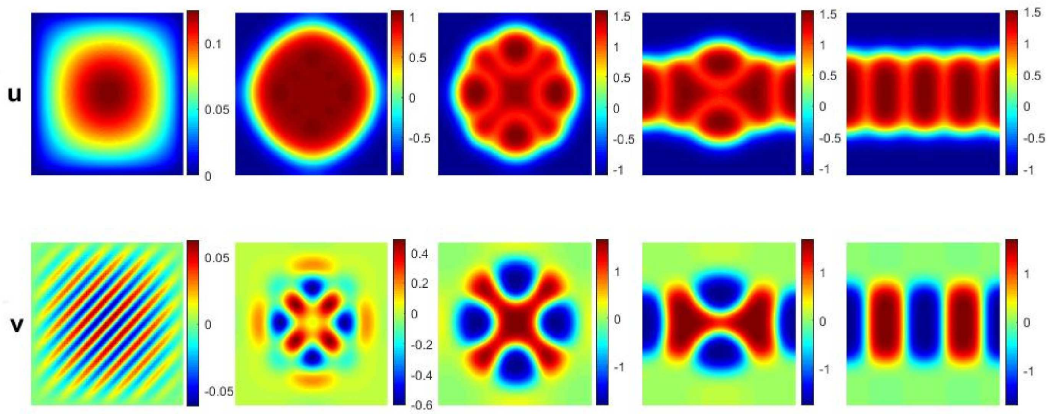


Fig. 4.7. Evolution of u (top) and v (bottom) with $\tau = 1e-04$ for solving the problem with initial data (4.3). From left to right: $t = 0, 0.3, 0.8, 37, 50$.

5. Conclusions

In this paper, we present a linearly implicit scheme for solving coupled Cahn-Hilliard system by using leap-frog method and the scalar auxiliary variable approach. The scheme is decoupled, unconditionally energy-stable and mass-conserving. It is shown that the scheme has second-order temporal accuracy. The error estimate and the boundedness of the numerical solution are given by using the mathematical induction method. Numerical experiments are presented to confirm the theoretical results.

Acknowledgments. The study was supported by the National Natural Science Foundation of China (Grant Nos. 12171442, 12231003, 12271215, 12326378, 11871248).

References

[1] G. Akrivis, B. Li, and D. Li, Energy-decaying extrapolated RK-SAV methods for the Allen-Cahn and Cahn-Hilliard equations, *SIAM J. Sci. Comput.*, **41**:6 (2019), A3703–A3727.

[2] E. Avalos, T. Higuchi, T. Teramoto, H. Yabu, and Y. Nishiura, Frustrated phases under three-dimensional confinement simulated by a set of coupled Cahn-Hilliard equations, *Soft Matter*, **12**:40 (2016), 5905–5914.

[3] E. Avalos, T. Teramoto, H. Komiyama, H. Yabu, and Y. Nishiura, Transformation of block copolymer nanoparticles from ellipsoids with striped lamellae into onionlike spheres and dynamical control via coupled Cahn-Hilliard equations, *ACS Omega*, **3**:1 (2018), 1304–1314.

[4] S. Badia, F. Guillén-González, and J.V. Gutiérrez-Santacreu, Finite element approximation of nematic liquid crystal flows using a saddle-point structure, *J. Comput. Phys.*, **230**:4 (2011), 1686–1706.

[5] Q. Cheng, C. Liu, and J. Shen, A new Lagrange multiplier approach for gradient flows, *Comput. Methods Appl. Mech. Engrg.*, **367** (2020), 113070.

[6] Q. Cheng, C. Liu, and J. Shen, Generalized SAV approaches for gradient systems, *J. Comput. Appl. Math.*, **394** (2021), 113532.

[7] L. Cherfils and A. Miranville, On a coupled Cahn-Hilliard system for copolymer/homopolymer mixtures, *J. Math. Study*, **55**:1 (2022), 22–37.

[8] A. Di Primio and M. Grasselli, On a system of coupled Cahn-Hilliard equations, *Nonlinear Anal. Real World Appl.*, **67** (2022), 103601.

[9] C.M. Elliott and A.M. Stuart, The global dynamics of discrete semilinear parabolic equations, *SIAM J. Numer. Anal.*, **30**:6 (1993), 1622–1663.

[10] D.J. Eyre, Unconditionally gradient stable time marching the Cahn-Hilliard equation, in: *Materials Research Society Symposium Proceedings*, **529** (1998), 39–46.

[11] Y. Gong, J. Zhao, and Q. Wang, Arbitrarily high-order unconditionally energy stable SAV schemes for gradient flow models, *Comput. Phys. Commun.*, **249** (2020), 107033.

[12] F. Guillén-González and G. Tierra, On linear schemes for a Cahn-Hilliard diffuse interface model, *J. Comput. Phys.*, **234** (2013), 140–171.

[13] D.I. Ketcheson, Relaxation Runge-Kutta methods: Conservation and stability for inner-product norms, *SIAM J. Numer. Anal.*, **57**:6 (2019), 2850–2870.

[14] D. Li, X. Li, and H. Sun, Optimal error estimates of SAV Crank-Nicolson finite element method for the coupled nonlinear Schrödinger equation, *J. Sci. Comput.*, **97**:71 (2023), 1–26.

[15] D. Li, X. Li, and Z. Zhang, Linearly implicit and high-order energy-preserving relaxation schemes for highly oscillatory Hamiltonian systems, *J. Comput. Phys.*, **477** (2023), 111925.

[16] D. Li, X. Li, and Z. Zhang, Implicit-explicit relaxation Runge-Kutta methods: Construction, analysis, and applications to PDES, *Math. Comp.*, **92**:339 (2023), 117–146.

[17] D. Li, Z. Qiao, and T. Tang, Characterizing the stabilization size for semi-implicit Fourier-spectral method to phase field equations, *SIAM J. Numer. Anal.*, **54**:3 (2016), 1653–1681.

[18] D. Li and W. Sun, Linearly implicit and high-order energy-conserving schemes for nonlinear wave equations, *J. Sci. Comput.*, **83**:65 (2020), 1–17.

[19] L. Li, Y. Yan, F. Yang, and X. Ma, Novel, linear, decoupled and unconditionally energy stable numerical methods for the coupled Cahn-Hilliard equations, *J. Inequal. Appl.*, **2022** (2022), 71.

[20] Q. Li and L. Mei, Efficient, decoupled, and second-order unconditionally energy stable numerical schemes for the coupled Cahn-Hilliard system in copolymer/homopolymer mixtures, *Comput. Phys. Commun.*, **260** (2021), 107290.

[21] S. Li and C. Liu, Global existence and decay estimates to a coupled Cahn-Hilliard system on $\mathcal{R}^N$, *J. Differential Equations*, **360** (2023), 151–181.

[22] X. Li, Z. Qiao, and C. Wang, Convergence analysis for a stabilized linear semi-implicit numerical scheme for the nonlocal Cahn-Hilliard equation, *Math. Comp.*, **90**:327 (2020), 171–188.

[23] X. Li, J. Wen, and D. Li, Mass- and energy-conserving difference schemes for nonlinear fractional Schrödinger equations, *Appl. Math. Lett.*, **111** (2021), 106686.

[24] Y. Li, L. Zhang, Q. Xia, Q. Yu, and J. Kim, An unconditionally energy-stable second-order time-accurate numerical scheme for the coupled Cahn-Hilliard system in copolymer/homopolymer mixtures, *Comput. Mater. Sci.*, **200** (2021), 110809.

[25] J. Shen and J. Xu, Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows, *SIAM J. Numer. Anal.*, **56**:5 (2018), 2895–2912.

[26] J. Shen, J. Xu, J. Yang, The scalar auxiliary variable (SAV) approach for gradient flows, *J. Comput. Phys.*, **353** (2018), 407–416.

[27] J. Shen and X. Yang, Numerical approximations of Allen-Cahn and Cahn-Hilliard equations, *Discrete Contin. Dyn. Syst.*, **28**:4 (2010), 1669.

[28] T. Tang, X. Wu, and J. Yang, Arbitrarily high order and fully discrete extrapolated RK-SAV/DG schemes for phase-field gradient flows, *J. Sci. Comput.*, **93**:2 (2022), 38.

[29] L. Wang and H. Yu, On efficient second order stabilized semi-implicit schemes for the Cahn-Hilliard phase-field equation, *J. Sci. Comput.*, **77**:2 (2018), 1185–1209.

[30] J. Wu, X. Zhang, and Z. Tan, An unconditionally energy stable algorithm for copolymer-homopolymer mixtures, *Int. J. Mech. Sci.*, **238** (2023), 107846.

[31] J. Yang and J. Kim, Linear, second-order accurate, and energy stable scheme for a ternary Cahn-Hilliard model by using Lagrange multiplier approach, *Acta Appl. Math.*, **172**:1 (2021), 10.

[32] X. Yang and L. Ju, Efficient linear schemes with unconditional energy stability for the phase field elastic bending energy model, *Comput. Methods Appl. Mech. Engrg.*, **315** (2017), 691–712.

[33] X. Yang and G. Zhang, Convergence analysis for the invariant energy quadratization (IEQ) schemes for solving the Cahn-Hilliard and Allen-Cahn equations with general nonlinear potential, *J. Sci. Comput.*, **82**:55 (2020), 1–28.

[34] X. Yang and X. He, A fully-discrete decoupled finite element method for the conserved Allen-Cahn type phase-field model of three-phase fluid flow system, *Comput. Methods Appl. Mech. Engrg.*, **389** (2022), 114376.

[35] X. Yang and X. He, Numerical approximations of flow coupled binary phase field crystal system: Fully discrete finite element scheme with second-order temporal accuracy and decoupling structure, *J. Comput. Phys.*, **467** (2022), 111448.

[36] X. Yang, J. Zhao, Q. Wang, and J. Shen, Numerical approximations for a three-component Cahn-Hilliard phase-field model based on the invariant energy quadratization method, *Math. Models Methods Appl. Sci.*, **27**:11 (2017), 1993–2030.

[37] Q. Zhang, T. Yan, and G. Gao, The energy method for high-order invariants in shallow water wave equations, *Appl. Math. Lett.*, **142** (2023), 108626.