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A New Multiphysics Finite Element Method for a Biot Model with Secondary Consolidation

Zhihao Ge* and Wenlong He

School of Mathematics and Statistics, Henan University, Kaifeng 475004, P.R. China.

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Abstract. In this paper, we propose a new multiphysics finite element method for a Biot model with secondary consolidation in soil dynamics. To better describe the multiphysical processes underlying in the original model and propose stable numerical methods to overcome "locking phenomenon" of pressure and displacement, we reformulate the swelling clay model with secondary consolidation by a new multiphysics approach, which transforms the fluid-solid coupling problem to a fluid coupled problem. Then, we give the energy law and prior error estimate of the weak solution. Also, we design a fully discrete time-stepping scheme to use multiphysics finite element method with $P_2-P_1-P_1$ element pairs for the space variables and backward Euler method for the time variable, and we derive the discrete energy laws and the optimal convergence order error estimates. Also, we show some numerical examples to verify the theoretical results and there is no "locking phenomenon". Finally, we draw conclusions to summarize the main results of this paper.

AMS subject classifications: 65N30, 65N12

Key words: Biot model, multiphysics finite element method, optimal convergence order, secondary consolidation.

1 Introduction

Biot model with secondary consolidation in soil dynamics plays a very important role in the construction of civil engineering, such as industrial and civil buildings, roads and bridges, water conservancy facilities, embankments and ports (cf. [2, 3, 18, 21, 30]). Compression deformation of saturated clays is usually based on Terzaghi's consolidation theory and Biot's consolidation(cf. [2, 26, 30]), which are the primary consolidation theories.

*Corresponding author. *Email addresses:* zhihaoge@henu.edu.cn (Z. Ge), hwl716825@163.com (W. He)

However, secondary consolidation is a process in which the volume of saturated clay decreases with time after the completion of primary consolidation, which plays an important role in the study of clay. The control equations of Biot model with secondary consolidation are the same as ones of general poroelasticity model. And the general poroelasticity model is widely applied in various fields such as geophysics, biomechanics, chemical engineering, materials science and so on, one can refer to [2,7,9,13,15,17,22,23,31]. In this paper, we consider the following Biot model with secondary consolidation (quasi-static poroelasticity model, cf. [28]):

$$-\lambda^*\nabla(\mathrm{div}\boldsymbol{\tau})_t-\mathrm{div}\sigma(\boldsymbol{\tau})+b_0\nabla p=\mathbf{F} \quad \text{in } \Omega_T:=\Omega\times(0,T)\subset\mathbb{R}^d\times(0,T), \tag{1.1}$$

$$(a_0p+b_0\mathrm{div}\boldsymbol{\tau})_t+\mathrm{div}\boldsymbol{\zeta}_f=\phi \quad \text{in } \Omega_T, \tag{1.2}$$

where

$$\sigma(\boldsymbol{\tau})=\gamma\varepsilon(\boldsymbol{\tau})+\beta tr\big(\varepsilon(\boldsymbol{\tau})\big)\mathbf{I}, \quad \varepsilon(\boldsymbol{\tau})=\frac{1}{2}\big(\nabla\boldsymbol{\tau}+(\nabla\boldsymbol{\tau})'\big), \tag{1.3}$$

$$\boldsymbol{\zeta}_f:=-\frac{K}{\theta_f}(\nabla p-\rho_f\mathbf{g}). \tag{1.4}$$

Here $\Omega\subset\mathbb{R}^d\,(d=1,2,3)$ denotes a bounded polygonal domain with the boundary $\partial\Omega$. $\boldsymbol{\tau}$ denotes the displacement vector of the solid, p denotes the pressure of the solvent, $\mathbf{I}$ denotes the $d\times d$ identity matrix, $\mathbf{F}$ is the body force. The permeability tensor $K=K(x)$ is assumed to be symmetric and uniformly positive definite in the sense that there exists positive constants K_1 and K_2 such that $K_1|\zeta|^2\le K(x)\zeta\cdot\zeta\le K_2|\zeta|^2$ for a.e. $x\in\Omega$ and $\zeta\in\mathbb{R}^d$. The fluid viscosity θ_f, Biot-Willis constant b_0, secondary consolidation coefficient λ^* and the constrained specific storage coefficient a_0 are all non-negative. In addition, $\tilde{\sigma}(\boldsymbol{\tau})$ is called the (effective) stress tensor. $\boldsymbol{\zeta}_f$ is the volumetric fluid flux, which is called Darcy's law. β and γ are Lamé constants, $\widehat{\sigma}(\boldsymbol{\tau},p):=\sigma(\boldsymbol{\tau})-b_0p\mathbf{I}$ is the total stress tensor. We assume that $\rho_f\not\equiv0$, which is a realistic assumption.

As for Biot model with primary consolidation, Phillips and Wheeler [24] propose and analyze a continuous-in-time linear poroelasticity model. Feng *et al.* [11] propose a multiphysics approach to reformulate the linear poroelasticity model to propose a stable finite element method. The second consolidation is introduced and developed by Cushman and Murad [21]. Showalter [28] finds that the term of $\lambda^*\nabla(\mathrm{div}\boldsymbol{\tau})_t$ has an effect for the momentum equation (1.1) when $\lambda^*>0$ just as the effect for the diffusion equation (1.2) when $a_0>0$. Gaspar [14] introduces a stabilized method for a Biot model with secondary consolidation by using the finite difference method on staggered grids. Lewis and Schrefler [19] use the finite element method to study the Biot model with secondary consolidation but not overcome the "locking phenomenon". In this paper, following the idea of [11], we reformulate the Biot model with secondary consolidation into a fluid coupled problem. A new breakthrough has been made by introducing new variables which are very different from ones of [11] (see Remark 2.1), that is, by introducing the auxiliary variable $q=\mathrm{div}\boldsymbol{\tau}$ and new variables $\varpi=a_0p+b_0q,\delta=b_0p-\beta q-\lambda^*q_t$, we reformulate the problem (1.1)-(1.2) into a generalized Stokes problem (when $\beta\to+\infty$) coupled with a diffusion problem, so it has some challenge in PDE analysis and numerical analysis. Also, we give the PDE analysis of the original and reformulated problem, propose a fully discrete multiphysics finite element method and prove that the proposed method has an optimal convergence order, which has a built-in mechanism to overcomes the "locking phenomenon" of pressure and displacement.

The remainder of this paper is organized as follows. In Section 2, we reformulate the original model based on a new multiphysics approach to a fluid-fluid coupling system and give the definition of weak solution to the original model and the reformulated model. And we give the energy laws and prior error estimates. In Section 3, we propose and analyze the coupled and decoupled time stepping methods based on the multiphysics approach and prove that the time-stepping has an optimal convergence order. In Section 4, we provide some numerical experiments to verify the theoretical results of the proposed methods. Finally, we draw conclusions to summarize the main results of this paper.

2 Multiphysics reformulation and PDE analysis

To close the above system, we set the following boundary and initial conditions in this paper:

$$\widehat{\sigma}(\boldsymbol{\tau},p)\mathbf{n}=\lambda^*(\mathrm{div}\boldsymbol{\tau})_t\mathbf{n}+\sigma(\boldsymbol{\tau})\mathbf{n}-b_0p\mathbf{n}=\mathbf{F}_1 \quad \text{on } \partial\Omega_T=\partial\Omega\times(0,T), \tag{2.1}$$

$$\boldsymbol{\zeta}_f\cdot\mathbf{n}=-\frac{K}{\theta_f}(\nabla p-\rho_f\mathbf{g})\cdot\mathbf{n}=\phi_1 \quad \text{on } \partial\Omega_T, \tag{2.2}$$

$$\boldsymbol{\tau}=\boldsymbol{\tau}_0,\quad p=p_0 \quad \text{on } \Omega\times\{t=0\}. \tag{2.3}$$

In order to reveal the muti-physical processes underlying in the original model and propose stable numerical methods to overcome "locking phenomenon" of pressure and displacement, we introduce new variables

$$q=\mathrm{div}\boldsymbol{\tau},\quad \varpi=a_0p+b_0q,\quad \delta=b_0p-\beta q-\lambda^*q_t.$$

In some engineering literature, Lamé constant γ is also called the shear modulus and denoted by G, and $B:=\beta+2G/3$ is called the bulk modulus. The terms β,γ and B are computed from the Young's modulus E and the Poisson ratio ν by the following formulas:

$$\beta=\frac{E\nu}{(1+\nu)(1-2\nu)},\quad \gamma=G=\frac{E}{2(1+\nu)},\quad B=\frac{E}{3(1-2\nu)}.$$

It is easy to check that

$$p=\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1q_t,\quad q=\chi_1\varpi-\chi_3\delta-\lambda^*\chi_3q_t, \tag{2.4}$$

where

$$\chi_1=\frac{b_0}{b_0^2+\beta a_0},\quad \chi_2=\frac{\beta}{b_0^2+\beta a_0},\quad \chi_3=\frac{a_0}{b_0^2+\beta a_0}.$$

Then the problem (1.1)-(1.4) can be rewritten as

$$-\gamma\mathrm{div}\,\varepsilon(\boldsymbol{\tau})+\nabla\delta=\mathbf{F} \qquad \text{in } \Omega_T, \tag{2.5}$$

$$\chi_3\delta+\mathrm{div}\boldsymbol{\tau}+\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_t=\chi_1\varpi \qquad \text{in } \Omega_T, \tag{2.6}$$

$$\varpi_t-\frac{1}{\theta_f}\mathrm{div}\left[K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1 q_t)-\rho_f\mathbf{g}\big)\right]=\phi \qquad \text{in } \Omega_T. \tag{2.7}$$

The boundary and initial conditions (2.1)-(2.3) can be rewritten as

$$\lambda^*(\mathrm{div}\boldsymbol{\tau})_t\mathbf{n}+\sigma(\boldsymbol{\tau})\mathbf{n}-b_0(\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1 q_t)\mathbf{n}=\mathbf{F}_1 \qquad \text{on } \partial\Omega_T:=\partial\Omega\times(0,T), \tag{2.8}$$

$$-\frac{K}{\theta_f}\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1 q_t)-\rho_f\mathbf{g}\big)\cdot\mathbf{n}=\phi_1 \qquad \text{on } \partial\Omega_T, \tag{2.9}$$

$$\boldsymbol{\tau}=\boldsymbol{\tau}_0,\quad p=p_0 \qquad \text{in } \Omega\times\{t=0\}. \tag{2.10}$$

Remark 2.1. Since the diffusion process is underlying in the original model, by introducing the auxiliary variable q and new variables ϖ and δ, the original model is reformulated into a generalized Stokes problem coupled with a diffusion problem. The reformulated problem also has a significant advantage comparing with the original model because it will be helpful to design some stable finite element methods(the generalized Stokes problem can be solved by any stable Stokes solver and the diffusion problem can be solved by any Lagrange element) to overcome the "locking phenomenon" (see Test 3) when using the standard Galerkin mixed finite element method to solve the problem (1.1)-(1.2).

The standard function space notations are adopted in this paper, their precise definitions can be found in [4,6,29]. In particular, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote respectively the standard $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products. For any Banach space B, we let $\mathbf{B}=[B]^d$ and $\mathbf{B}'$ denote its dual space, and $\|\cdot\|_{L^p(B)}$ is a shorthand notation for $\|\cdot\|_{L^p((0,T);B)}$. We also introduce the function spaces

$$L_0^2(\Omega):=\{q\in L^2(\Omega):(q,1)=0\},\quad \mathbf{X}:=\mathbf{H}^1(\Omega).$$

From [29], it is well known that the following inf-sup condition holds in the space $\mathbf{X}\times L_0^2(\Omega)$:

$$\sup_{\mathbf{v}\in\mathbf{X}}\frac{(\mathrm{div}\mathbf{v},\varphi)}{\|\mathbf{v}\|_{H^1(\Omega)}}\geq\alpha_0\|\varphi\|_{L^2(\Omega)},\quad \forall\varphi\in L_0^2(\Omega),\quad \alpha_0>0. \tag{2.11}$$

Let

$$\mathbf{RM}:=\{\mathbf{r}=\mathbf{a}+\mathbf{b}\times x:\ \mathbf{a},\mathbf{b},x\in\mathbb{R}^d\}$$

denote the space of infinitesimal rigid motions. It is well known [4,16,29] that **RM** is the kernel of the strain operator ε, that is, $\mathbf{r}\in\mathbf{RM}$ if and only if $\varepsilon(\mathbf{r})=0$. Hence, we have

$$\varepsilon(\mathbf{r})=0,\quad \operatorname{div}\mathbf{r}=0,\quad \forall\mathbf{r}\in\mathbf{RM}. \tag{2.12}$$

Let $\mathbf{L}^2_\perp(\partial\Omega)$ and $\mathbf{H}^1_\perp(\Omega)$ denote respectively the subspaces of $\mathbf{L}^2(\partial\Omega)$ and $\mathbf{H}^1(\Omega)$ which are orthogonal to **RM**, that is,

$$\begin{aligned}
&\mathbf{H}^1_\perp(\Omega):=\{\mathbf{v}\in\mathbf{H}^1(\Omega):(\mathbf{v},\mathbf{r})=0,\forall\mathbf{r}\in\mathbf{RM}\},\\
&\mathbf{L}^2_\perp(\partial\Omega):=\{\mathbf{g}\in\mathbf{L}^2(\partial\Omega):\langle\mathbf{g},\mathbf{r}\rangle=0,\forall\mathbf{r}\in\mathbf{RM}\}.
\end{aligned}$$

It is well known [8] that there exists a constant $c_1>0$ such that

$$\inf_{\mathbf{r}\in\mathbf{RM}}\|\mathbf{v}+\mathbf{r}\|_{L^2(\Omega)}\le c_1\|\varepsilon(\mathbf{v})\|_{L^2(\Omega)},\quad \forall\mathbf{v}\in\mathbf{H}^1(\Omega). \tag{2.13}$$

From [11], we know that for each $\mathbf{v}\in\mathbf{H}^1_\perp(\Omega)$ there holds the following inf-sup condition:

$$\sup_{\mathbf{v}\in\mathbf{H}^1_\perp(\Omega)}\frac{(\operatorname{div}\mathbf{v},\varphi)}{\|\mathbf{v}\|_{H^1(\Omega)}}\ge\alpha_1\|\varphi\|_{L^2(\Omega)},\quad \forall\varphi\in L^2_0(\Omega),\quad \alpha_1>0. \tag{2.14}$$

Remark 2.2. The reason of introducing the space $\mathbf{H}^1_\perp(\Omega)$ is that the boundary condition (2.1) is a pure Neumann condition. If it is replaced by a pure Dirichlet condition or by a mixed Dirichlet-Neumann condition, there is no need to introduce this space.

For convenience, we assume that $\mathbf{F},\mathbf{F}_1,\phi$ and ϕ_1 all are independent of t in the remaining of the paper. We note that all the results of this paper can be easily extended to the case of time-dependent source functions.

Definition 2.1. *Let* $\boldsymbol{\tau}_0\in\mathbf{H}^1(\Omega),\mathbf{F}\in\mathbf{L}^2(\Omega),\mathbf{F}_1\in\mathbf{L}^2(\partial\Omega),p_0\in L^2(\Omega),\phi\in L^2(\Omega)$, *and* $\phi_1\in L^2(\partial\Omega)$. *Assume* $a_0>0$ *and* $(\mathbf{F},\mathbf{v})+\langle\mathbf{F}_1,\mathbf{v}\rangle=0$ *for any* $\mathbf{v}\in\mathbf{RM}$. *Given* $T>0$, *a tuple* $(\boldsymbol{\tau},p)$ *with*

$$\begin{aligned}
&\boldsymbol{\tau}\in L^\infty\big(0,T;\mathbf{H}^1_\perp(\Omega)\big),\\
&p\in L^\infty\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H^1(\Omega)\big),\\
&p_t,(\operatorname{div}\boldsymbol{\tau})_t\in L^2\big(0,T;H^1(\Omega)'\big)
\end{aligned}$$

is called a weak solution to the problem (1.1)-(1.2) *with* (2.1)-(2.3), *if there hold for almost every* $t\in[0,T]$

$$\begin{aligned}
&\lambda^*\big((\operatorname{div}\boldsymbol{\tau})_t,\operatorname{div}\mathbf{v}\big)+\gamma\big(\varepsilon(\boldsymbol{\tau}),\varepsilon(\mathbf{v})\big)+\beta(\operatorname{div}\boldsymbol{\tau},\operatorname{div}\mathbf{v})-b_0(p,\operatorname{div}\mathbf{v})\\
&=(\mathbf{F},\mathbf{v})+\langle\mathbf{F}_1,\mathbf{v}\rangle,\quad \forall\mathbf{v}\in\mathbf{H}^1(\Omega), &(2.15)\\
&\big((a_0p+b_0\operatorname{div}\boldsymbol{\tau})_t,\varphi\big)+\frac{1}{\theta_f}\big(K(\nabla p-\rho_f\mathbf{g}),\nabla\varphi\big)\\
&=(\phi,\varphi)+\langle\phi_1,\varphi\rangle,\quad \forall\varphi\in H^1(\Omega), &(2.16)\\
&\boldsymbol{\tau}(0)=\boldsymbol{\tau}_0,\quad p(0)=p_0. &(2.17)
\end{aligned}$$

Similarly, we can define the weak solution to the problem (2.5)-(2.10) as follows.

Definition 2.2. *Let* $\boldsymbol{\tau}_0\in\mathbf{H}^1(\Omega),\mathbf{F}\in\mathbf{L}^2(\Omega),\mathbf{F}_1\in\mathbf{L}^2(\partial\Omega),p_0\in L^2(\Omega),\phi\in L^2(\Omega)$, *and* $\phi_1\in L^2(\partial\Omega)$. *Assume* $a_0>0$ *and* $(\mathbf{F},\mathbf{v})+\langle\mathbf{F}_1,\mathbf{v}\rangle=0$ *for any* $\mathbf{v}\in\mathbf{RM}$. *Given* $T>0$, *a 3-tuple* $(\boldsymbol{\tau},\delta,\varpi)$ *with*

$$
\begin{aligned}
&\boldsymbol{\tau}\in L^\infty\big(0,T;\mathbf{H}^1_\perp(\Omega)\big), && \delta\in L^\infty\big(0,T;L^2(\Omega)\big),\\
&\varpi\in L^\infty\big(0,T;L^2(\Omega)\big)\cap H^1\big(0,T;H^1(\Omega)'\big), && q\in L^\infty\big(0,T;L^2(\Omega)\big),\\
&p\in L^\infty\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H^1(\Omega)\big)
\end{aligned}
$$

is called a weak solution to the problem (2.5)-(2.10), *if there hold for almost every* $t\in[0,T]$

$$
\begin{aligned}
&\gamma\big(\varepsilon(\boldsymbol{\tau}),\varepsilon(\mathbf{v})\big)-(\delta,\mathrm{div}\mathbf{v})=(\mathbf{F},\mathbf{v})+\langle\mathbf{F}_1,\mathbf{v}\rangle, && \forall\mathbf{v}\in\mathbf{H}^1(\Omega), && (2.18)\\
&\chi_3(\delta,\varphi)+(\mathrm{div}\boldsymbol{\tau},\varphi)+\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_t,\varphi)=\chi_1(\varpi,\varphi), && \forall\varphi\in L^2(\Omega), && (2.19)\\
&(\varpi_t,\psi)+\frac{1}{\theta_f}\big(K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1\mathrm{div}\boldsymbol{\tau}_t)-\rho_f\mathbf{g}\big),\nabla\psi\big) \\
=\;&(\phi,\psi)+\langle\phi_1,\psi\rangle, && \forall\psi\in H^1(\Omega), && (2.20)\\
&p:=\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1\mathrm{div}\boldsymbol{\tau}_t,\quad q:=\chi_1\varpi-\chi_3\delta-\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_t, &&&& (2.21)\\
&\varpi(0)=\varpi_0:=a_0p_0+b_0q_0, &&&& (2.22)
\end{aligned}
$$

where $q_0:=\mathrm{div}\boldsymbol{\tau}_0$, u_0 *and* p_0 *are same as in Definition* 2.1.

Lemma 2.1. *Every weak solution* $(\boldsymbol{\tau},\delta,\varpi)$ *of the problem* (2.18)-(2.22) *satisfies the following energy law:*

$$
\begin{aligned}
&J(s)+\lambda^*\int_0^s\|(\mathrm{div}\boldsymbol{\tau})_t\|^2_{L^2(\Omega)}dt\\
&\quad+\frac{1}{\theta_f}\int_0^s\big(K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*q_t)-\rho_f\mathbf{g}\big),\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*q_t)\big)dt\\
&\quad-\int_0^s(\phi,\chi_1\delta+\chi_2\varpi+\lambda^*q_t)dt-\int_0^s\langle\phi_1,\chi_1\delta+\chi_2\varpi+\lambda^*q_t\rangle dt=J(0)
\end{aligned}
\tag{2.23}
$$

for all $s\in[0,T]$, *where*

$$
\begin{aligned}
J(s)=\frac{1}{2}\Big[&\gamma\|\varepsilon\big(\boldsymbol{\tau}(s)\big)\|^2_{L^2(\Omega)}+\chi_2\|\varpi(s)\|^2_{L^2(\Omega)}+\chi_3\|\delta(s)\|^2_{L^2(\Omega)}\\
&+(\lambda^*)^2\chi_3\|\mathrm{div}\boldsymbol{\tau}_t(s)\|^2_{L^2(\Omega)}+\lambda^*\chi_3\big(\mathrm{div}\boldsymbol{\tau}_t(s),\delta(s)\big)-2\big(\mathbf{F},\boldsymbol{\tau}(s)\big)-2\langle\mathbf{F}_1,\boldsymbol{\tau}(s)\rangle\Big].
\end{aligned}
$$

Moreover, there holds

$$
\begin{aligned}
\|\varpi_t\|_{L^2(0,T;H^1(\Omega)')}\le&\frac{1}{\theta_f}\|K\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*q_t)-\rho_f\mathbf{g}\|_{L^2(\Omega)}\\
&+\|\phi\|_{L^2(\Omega)}+\|\phi_1\|_{L^2(\partial\Omega)}<\infty.
\end{aligned}
\tag{2.24}
$$

Proof. We only consider the case of $\boldsymbol{\tau}_t \in L^2(0,T;L^2(\Omega))$. Setting $\mathbf{v}=\boldsymbol{\tau}_t$ in (2.18), differentiating (2.19) with respect to t, taking $\varphi=\delta$ and setting $\psi=p=\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1\mathrm{div}\boldsymbol{\tau}_t$ in (2.20), we have

$$\gamma\big(\varepsilon(\boldsymbol{\tau}),\varepsilon(\boldsymbol{\tau}_t)\big)-(\delta,\mathrm{div}\boldsymbol{\tau}_t)=(\mathbf{F},\boldsymbol{\tau}_t)+\langle\mathbf{F}_1,\boldsymbol{\tau}_t\rangle, \tag{2.25}$$

$$\chi_3(\delta_t,\delta)+(\mathrm{div}\boldsymbol{\tau}_t,\delta)+\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta)=\chi_1\big(\varpi_t,\delta\big), \tag{2.26}$$

$$(\varpi_t,p)+\frac{1}{\theta_f}\big(K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*\chi_1\mathrm{div}\boldsymbol{\tau}_t)-\rho_f\mathbf{g}\big),\nabla p\big)=(\phi,p)+\langle\phi_1,p\rangle. \tag{2.27}$$

Adding the resulting equations and integrating in t from 0 to s, we get

$$\begin{aligned}
&\frac{\gamma}{2}\|\varepsilon\big(\boldsymbol{\tau}(s)\big)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi(s)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\delta(s)\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{\theta_f}\int_0^s\big(K(\nabla p-\rho_f\mathbf{g}),\nabla p\big)\,dt-\int_0^s(\phi,p)\,dt-\int_0^s\langle\phi_1,p\rangle\,dt\\
&\quad-\big(\mathbf{F},\boldsymbol{\tau}(s)\big)-\langle\mathbf{F}_1,\boldsymbol{\tau}(s)\rangle+\int_0^s\big[\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta)+\lambda^*\chi_1(\varpi_t,\mathrm{div}\boldsymbol{\tau}_t)\big]\,dt\\
=&\frac{\gamma}{2}\|\varepsilon\big(\boldsymbol{\tau}(0)\big)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi(0)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\delta(0)\|^2_{L^2(\Omega)}\\
&\quad-\big(\mathbf{F},\boldsymbol{\tau}(0)\big)-\langle\mathbf{F}_1,\boldsymbol{\tau}(0)\rangle.
\end{aligned} \tag{2.28}$$

Using the equality $q_t=\chi_1\varpi_t-\chi_3\delta_t-\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_{tt}$, we have

$$\begin{aligned}
&\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta)+\lambda^*\chi_1(\varpi_t,\mathrm{div}\boldsymbol{\tau}_t)\\
=&\lambda^*\chi_3\left[\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_t,\delta)-(\mathrm{div}\boldsymbol{\tau}_t,\delta_t)\right]+\lambda^*\chi_1(\varpi_t,\mathrm{div}\boldsymbol{\tau}_t)\\
=&\lambda^*\chi_3\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_t,\delta)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_t,\chi_1\varpi_t-\chi_3\delta_t)\\
=&\lambda^*\chi_3\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_t,\delta)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_t,\mathrm{div}\boldsymbol{\tau}_t+\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_{tt})\\
=&\lambda^*\chi_3\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_t,\delta)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_t,\mathrm{div}\boldsymbol{\tau}_t)+\frac{(\lambda^*)^2\chi_3}{2}\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_t,\mathrm{div}\boldsymbol{\tau}_t).
\end{aligned} \tag{2.29}$$

Using (2.29) and (2.28), we get

$$\begin{aligned}
&\frac{\gamma}{2}\|\varepsilon\big(\boldsymbol{\tau}(s)\big)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi(s)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\delta(s)\|^2_{L^2(\Omega)}+\lambda^*\chi_3\big(\mathrm{div}\boldsymbol{\tau}_t(s),\delta(s)\big)\\
&\quad+\frac{1}{\theta_f}\int_0^s\big(K(\nabla p-\rho_f\mathbf{g}),\nabla p\big)\,dt-\int_0^s(\phi,p)\,dt-\int_0^s\langle\phi_1,p\rangle\,dt-\big(\mathbf{F},\boldsymbol{\tau}(s)\big)\\
&\quad-\langle\mathbf{F}_1,\boldsymbol{\tau}(s)\rangle+\int_0^s\lambda^*\|\mathrm{div}\boldsymbol{\tau}_t\|^2_{L^2(\Omega)}\,dt+\frac{(\lambda^*)^2\chi_3}{2}\|\mathrm{div}\boldsymbol{\tau}_t(s)\|^2_{L^2(\Omega)}\\
=&\frac{\gamma}{2}\|\varepsilon\big(\boldsymbol{\tau}(0)\big)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi(0)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\delta(0)\|^2_{L^2(\Omega)}-\big(\mathbf{F},\boldsymbol{\tau}(0)\big)
\end{aligned}$$

$$-\langle \mathbf{F}_1,\boldsymbol{\tau}(0)\rangle+\lambda^*\chi_3\big(\mathrm{div}\boldsymbol{\tau}_t(0),\delta(0)\big)+\frac{(\lambda^*)^2\chi_3}{2}\|\mathrm{div}\boldsymbol{\tau}_t(0)\|^2_{L^2(\Omega)}, \tag{2.30}$$

which implies that (2.23) holds.

Using (2.20), Cauchy-Schwarz inequality, the trace theorem (cf. [10]), we have

$$\begin{aligned}(\varpi_t,\psi)&=-\frac{1}{\theta_f}\big(K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*q_t)-\rho_f\mathbf{g}\big),\nabla\psi\big)+(\phi,\psi)+\langle\phi_1,\psi\rangle\\&\le\frac{1}{\theta_f}\big\|K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*q_t)-\rho_f\mathbf{g}\big)\big\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}\\&\quad+\|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}+\|\phi_1\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)}\\&\le\frac{1}{\theta_f}\big\|K\big(\nabla(\chi_1\delta+\chi_2\varpi+\lambda^*q_t)-\rho_f\mathbf{g}\big)\big\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}\\&\quad+\|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}+\|\phi_1\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\Omega)}.\end{aligned}\tag{2.31}$$

Using (2.31), Young inequality and the definition of the $H^1(\Omega)'$-norm, we see that (2.24) holds. The proof is complete. □

Taking the similar argument, we can prove that the weak solution of (2.22)-(2.23) satisfies the following energy law, here we omit the details of the proof.

Lemma 2.2. *Every weak solution* $(\boldsymbol{\tau},p)$ *of the problem* (2.15)-(2.17) *satisfies the following energy law:*

$$\begin{aligned}&E(s)+\lambda^*\int_0^s\|(\mathrm{div}\boldsymbol{\tau})_t\|^2_{L^2(\Omega)}dt+\frac{1}{\theta_f}\int_0^s\big(K(\nabla p-\rho_f\mathbf{g}),\nabla p\big)dt\\&\quad-\int_0^s(\phi,p)dt-\int_0^s\langle\phi_1,p\rangle dt=E(0)\end{aligned}\tag{2.32}$$

for all $s\in[0,T]$, *where*

$$\begin{aligned}E(s):=\frac{1}{2}\Big[&\gamma\big\|\varepsilon\big(\boldsymbol{\tau}(s)\big)\big\|^2_{L^2(\Omega)}+\beta\|\mathrm{div}\boldsymbol{\tau}(s)\|^2_{L^2(\Omega)}\\&+a_0\|p(s)\|^2_{L^2(\Omega)}-2\big(\mathbf{F},\boldsymbol{\tau}(s)\big)-2\langle\mathbf{F}_1,\boldsymbol{\tau}(s)\rangle\Big].\end{aligned}\tag{2.33}$$

Moreover, there holds

$$\begin{aligned}&\|(a_0p+b_0\mathrm{div}\boldsymbol{\tau})_t\|_{L^2(0.T;H^1(\Omega)')}\\&\le\frac{1}{\theta_f}\|K\nabla p-\rho_f\mathbf{g}\|_{L^2(\Omega)}+\|\phi\|_{L^2(\Omega)}+\|\phi_1\|_{L^2(\partial\Omega)}<\infty.\end{aligned}\tag{2.34}$$

Lemma 2.3. *Every weak solution* $(\boldsymbol{\tau},\delta,\varpi,p,q)$ *to the problem* (2.22)-(2.23) *satisfies the following energy laws:*

$$C_\varpi(t):=\big(\varpi(\cdot,t),1\big)=(\varpi_0,1)+[(\phi,1)+\langle\phi_1,1\rangle]t,\quad t\ge0, \tag{2.35}$$

$$C_\delta(t):=\big(\delta(\cdot,t),1\big)=\big(\delta(\cdot,0),1\big)+\frac{\chi_1\gamma}{d+\chi_3\gamma}C_\varpi(t)-\frac{d\lambda^*\chi_3\chi_1\gamma}{(d+\chi_3\gamma)^2}[(\phi,1)+\langle\phi_1,1\rangle]$$
$$+\frac{1}{d+\chi_3\gamma}\big(-(\mathbf{F},\mathbf{x})-\langle\mathbf{F}_1,\mathbf{x}\rangle\big), \tag{2.36}$$

$$C_q(t):=\big(q(\cdot,t),1\big)=\chi_1C_\varpi(t)-\chi_3C_\delta(t), \tag{2.37}$$

$$C_p(t):=\big(p(\cdot,t),1\big)=\chi_1C_\delta(t)+\chi_2C_\varpi(t), \tag{2.38}$$

$$C_{\boldsymbol{\tau}}(t):=\langle\boldsymbol{\tau}(\cdot,t)\cdot\mathbf{n},1\rangle=C_q(t). \tag{2.39}$$

Proof. We first notice that (2.35) follows immediately from taking $\psi\equiv 1$ in (2.20). To prove (2.36), taking $\mathbf{v}=\mathbf{x}$ in (2.18) and $\varphi=1$ in (2.19), and using the identities $\nabla\mathbf{x}=\mathbf{I},\operatorname{div}\mathbf{x}=d$, and $\varepsilon(\mathbf{x})=\mathbf{I}$, we get

$$\begin{aligned}&\gamma\big(\varepsilon(\boldsymbol{\tau}),\mathbf{I}\big)=\gamma(\operatorname{div}\boldsymbol{\tau},1)=d(\delta,1)+(\mathbf{F},\mathbf{x})+\langle\mathbf{F}_1,\mathbf{x}\rangle,\\&(\operatorname{div}\boldsymbol{\tau},1)=\chi_1(\varpi,1)-\chi_3(\delta,1)-\lambda^*\chi_3(\operatorname{div}\boldsymbol{\tau}_t,1).\end{aligned}$$

It is easy to check that

$$d\lambda^*\chi_3(\delta_t,1)+(d+\chi_3\gamma)(\delta,1)=\chi_1\gamma(\varpi,1)-(\mathbf{F},\mathbf{x})-\langle\mathbf{F}_1,\mathbf{x}\rangle. \tag{2.40}$$

Applying the ordinary differential equation theory to (2.40), we have

$$\begin{aligned}(\delta,1)=&e^{-\int\frac{d+\chi_3\gamma}{d\lambda^*\chi_3}dt}\left(\int e^{\int\frac{d+\chi_3\gamma}{d\lambda^*\chi_3}dt}\left(\frac{\chi_1\gamma}{d\lambda^*\chi_3}(\varpi,1)-\frac{1}{d\lambda^*\chi_3}(\mathbf{F},\mathbf{x})-\frac{1}{d\lambda^*\chi_3}\langle\mathbf{F}_1,\mathbf{x}\rangle\right)dt+C\right)\\=&\frac{d\lambda^*\chi_3\chi_1\gamma}{(d+\chi_3\gamma)d\lambda^*\chi_3}(\varpi,1)-\left(\frac{d\lambda^*\chi_3}{d+\chi_3\gamma}\right)^2\frac{\chi_1\gamma}{d\lambda^*\chi_3}[(\phi,1)+\langle\phi_1,1\rangle]\\&+e^{-\frac{d+\chi_3\gamma}{d\lambda^*\chi_3}t}\left(\int e^{\frac{d+\chi_3\gamma}{d\lambda^*\chi_3}t}\left(-\frac{1}{d\lambda^*\chi_3}(\mathbf{F},\mathbf{x})-\frac{1}{d\lambda^*\chi_3}\langle\mathbf{F}_1,\mathbf{x}\rangle\right)dt+C\right),\end{aligned}$$

which implies that (2.36) holds.

Finally, using $q=\chi_1\varpi-\chi_3\delta$, $p=\chi_1\delta+\chi_2\varpi$, (2.35) and (2.36), we see that (2.37) and (2.38) hold. And (2.39) is an immediate consequence by applying Gauss divergence theorem to $\langle\boldsymbol{\tau}(\cdot,t)\cdot\mathbf{n},1\rangle=(\operatorname{div}\boldsymbol{\tau},1)$. The proof is complete. □

Using Lemmas 2.1 and 2.2, we have the following stability estimates.

Lemma 2.4. *There exist positive constants*

$$\begin{aligned}&\acute{C}_1=\acute{C}_1\big(\|\boldsymbol{\tau}_0\|_{H^1(\Omega)},\|p_0\|_{L^2(\Omega)},\|\mathbf{F}\|_{L^2(\Omega)},\|\mathbf{F}_1\|_{L^2(\partial\Omega)},\|\phi\|_{L^2(\Omega)},\|\phi_1\|_{L^2(\partial\Omega)}\big),\\&\acute{C}_2=\acute{C}_2\big(\acute{C}_1,\|\nabla p_0\|_{L^2(\Omega)}\big)\end{aligned}$$

such that

$$
\begin{aligned}
&\sqrt{\lambda^*}\|(\mathrm{div}\boldsymbol{\tau})_t\|_{L^2(0,T;L^2(\Omega))}+\sqrt{\gamma}\|\varepsilon(\boldsymbol{\tau})\|_{L^\infty(0,T;L^2(\Omega))}+\sqrt{\chi_2}\|\varpi\|_{L^\infty(0,T;L^2(\Omega))}\\
&\quad+\sqrt{\chi_3}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_t+\delta\|_{L^\infty(0,T;L^2(\Omega))}+\sqrt{\frac{K_1}{\theta_f}}\|\nabla p\|_{L^2(0,T;L^2(\Omega))}\le \acute{C}_1, \qquad (2.41)\\
&\|\boldsymbol{\tau}\|_{L^\infty(0,T;L^2(\Omega))}\le \acute{C}_1, \quad \|p\|_{L^\infty(0,T;L^2(\Omega))}\le \acute{C}_2\Big(\chi_2^{\frac{1}{2}}+\chi_1\chi_3^{-\frac{1}{2}}\Big), \qquad (2.42)\\
&\|p\|_{L^2(0,T;L^2(\Omega))}\le \acute{C}_1, \quad \|\delta\|_{L^2(0,T;L^2(\Omega))}\le \acute{C}_2\chi_1^{-1}\Big(1+\chi_2^{\frac{1}{2}}\Big). \qquad (2.43)
\end{aligned}
$$

Proof. It is easy to check that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\delta,\delta)+\frac{d}{dt}(\lambda^*\mathrm{div}\boldsymbol{\tau}_t,\delta)+\frac{1}{2}\frac{d}{dt}(\lambda^*\mathrm{div}\boldsymbol{\tau}_t,\lambda^*\mathrm{div}\boldsymbol{\tau}_t)\\
=&\frac{1}{2}\frac{d}{dt}(\lambda^*\mathrm{div}\boldsymbol{\tau}_t+\delta,\lambda^*\mathrm{div}\boldsymbol{\tau}_t+\delta). \qquad (2.44)
\end{aligned}
$$

Using (2.30) and (2.44), we get

$$
\begin{aligned}
&\frac{\gamma}{2}\big\|\varepsilon\big(\boldsymbol{\tau}(s)\big)\big\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi(s)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_t(s)+\delta(s)\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{\theta_f}\int_0^s\big(K(\nabla p-\rho_f\mathbf{g}),\nabla p\big)\,dt-\int_0^s(\phi,p)\,dt-\int_0^s\langle\phi_1,p\rangle\,dt-\big(\mathbf{F},\boldsymbol{\tau}(s)\big)\\
&\quad-\langle\mathbf{F}_1,\boldsymbol{\tau}(s)\rangle+\int_0^s\lambda^*\|\mathrm{div}\boldsymbol{\tau}_t\|^2_{L^2(\Omega)}\,dt\\
=&\frac{\gamma}{2}\big\|\varepsilon\big(\boldsymbol{\tau}(0)\big)\big\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi(0)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_t(0)+\delta(0)\|^2_{L^2(\Omega)}\\
&\quad-\big(\mathbf{F},\boldsymbol{\tau}(0)\big)-\langle\mathbf{F}_1,\boldsymbol{\tau}(0)\rangle. \qquad (2.45)
\end{aligned}
$$

Using (2.45), we have

$$
\begin{aligned}
&\sqrt{\lambda^*}\|(\mathrm{div}\boldsymbol{\tau})_t\|_{L^2(0,T;L^2(\Omega))}+\sqrt{\gamma}\|\varepsilon(\boldsymbol{\tau})\|_{L^\infty(0,T;L^2(\Omega))}+\sqrt{\chi_2}\|\varpi\|_{L^\infty(0,T;L^2(\Omega))}\\
&\quad+\sqrt{\chi_3}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_t+\delta\|_{L^\infty(0,T;L^2(\Omega))}+\sqrt{\frac{K_1}{\theta_f}}\|\nabla p\|_{L^2(0,T;L^2(\Omega))}\\
\le& C\Big(\big\|\varepsilon\big(\boldsymbol{\tau}(0)\big)\big\|_{L^2(\Omega)}+\|\varpi(0)\|_{L^2(\Omega)}+\|\mathrm{div}\boldsymbol{\tau}_t(0)\|_{L^2(\Omega)}+\|\delta(0)\|_{L^2(\Omega)}\\
&\quad+\|\mathbf{F}\|_{L^2(\Omega)}+\|\mathbf{F}_1\|_{L^2(\partial\Omega)}+\|\phi\|_{L^2(\Omega)}+\|\phi_1\|_{L^2(\partial\Omega)}\Big), \qquad (2.46)
\end{aligned}
$$

which implies that (2.41) holds. It's easy to check that (2.42) holds from (2.41), (2.13) and the relation $p=\chi_1\delta+\chi_2\varpi+\lambda^*q_t$. We note that (2.43) follows from (2.41), (2.13), (2.36), (2.38) and the relation $p=\chi_1\delta+\chi_2\varpi+\lambda^*q_t$. The proof is complete. □

Theorem 2.1. *Suppose that $\boldsymbol{\tau}_0$ and p_0 are sufficiently smooth, then there exist positive constants $\acute{C}_2=\acute{C}_2(\acute{C}_1,\|\nabla p_0\|_{L^2(\Omega)}),\acute{C}_3=\acute{C}_3(\acute{C}_1,\acute{C}_2,\|\boldsymbol{\tau}_0\|_{H^2(\Omega)},\|p_0\|_{H^2(\Omega)})$ such that*

$$\begin{aligned}&\sqrt{\lambda^*}\|(\mathrm{div}\boldsymbol{\tau})_t\|_{L^2(0,T;L^2(\Omega))}+\sqrt{\gamma}\|\varepsilon(\boldsymbol{\tau}_t)\|_{L^2(0,T;L^2(\Omega))}+\sqrt{\chi_2}\|\varpi_t\|_{L^2(0,T;L^2(\Omega))}\\&\quad+\sqrt{\chi_3}\|\delta_t\|_{L^2(0,T;L^2(\Omega))}+\sqrt{\frac{K_1}{\theta_f}}\|\nabla p\|_{L^\infty(0,T;L^2(\Omega))}\\&\quad+\sqrt{\gamma\lambda^*\chi_3}\|\varepsilon(\boldsymbol{\tau}_t)\|_{L^\infty(0,T;L^2(\Omega))}+\lambda^*\sqrt{\chi_3}\|\mathrm{div}\boldsymbol{\tau}_t\|_{L^\infty(0,T;L^2(\Omega))}\le\acute{C}_2,\end{aligned}\tag{2.47}$$

$$\begin{aligned}&\sqrt{\lambda^*}\|(\mathrm{div}\boldsymbol{\tau})_{tt}\|_{L^2(0,T;L^2(\Omega))}+\sqrt{\gamma}\|\varepsilon(\boldsymbol{\tau}_t)\|_{L^\infty(0,T;L^2(\Omega))}+\sqrt{\chi_2}\|\varpi_t\|_{L^\infty(0,T;L^2(\Omega))}\\&\quad+\sqrt{\chi_3}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt}+\delta_t\|_{L^\infty(0,T;L^2(\Omega))}+\sqrt{\frac{K_1}{\theta_f}}\|\nabla p_t\|_{L^2(0,T;L^2(\Omega))}\le\acute{C}_3,\end{aligned}\tag{2.48}$$

$$\|\varpi_{tt}\|_{L^2(H^1(\Omega)')}\le\sqrt{\frac{K_2}{\theta_f}}\acute{C}_3.\tag{2.49}$$

Proof. To show (2.47), firstly differentiating (2.18) one time with respect to t and setting $\mathbf{v}=\boldsymbol{\tau}_t$, differentiating (2.19) one time with respect to t and setting $\varphi=\delta_t$, taking $\psi=p_t=\chi_1\delta_t+\chi_2\varpi_t+\lambda^*\chi_1\mathrm{div}\boldsymbol{\tau}_t$ in (2.20), we get

$$\gamma\|\varepsilon(\boldsymbol{\tau}_t)\|^2_{L^2(\Omega)}-(\mathrm{div}\boldsymbol{\tau}_t,\delta_t)=0,\tag{2.50}$$

$$\chi_3\|\delta_t\|^2_{L^2(\Omega)}+(\mathrm{div}\boldsymbol{\tau}_t,\delta_t)+\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)=\chi_1(\varpi_t,\delta_t),\tag{2.51}$$

$$\begin{aligned}&\chi_1(\varpi_t,\delta_t)+\chi_2\|\varpi_t\|^2_{L^2(\Omega)}+\lambda^*\chi_1(\varpi_t,\mathrm{div}\boldsymbol{\tau}_{tt})\\&\quad+\frac{1}{\theta_f}\big(K(\nabla p-\rho_f\mathbf{g}),\nabla p_t\big)=\frac{d}{dt}[(\phi,p)-\langle\phi_1,p\rangle].\end{aligned}\tag{2.52}$$

Adding (2.50)-(2.52) and integrating in t from 0 to s, we get

$$\begin{aligned}&\int_0^s\gamma\|\varepsilon(\boldsymbol{\tau}_t)\|^2_{L^2(\Omega)}+\chi_2\|\varpi_t\|^2_{L^2(\Omega)}+\chi_3\|\delta_t\|^2_{L^2(\Omega)}+\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)dt\\&\quad+\frac{K}{2\theta_f}\|\nabla p(s)\|^2_{L^2(\Omega)}+\int_0^s\lambda^*\chi_1(\varpi_t,\mathrm{div}\boldsymbol{\tau}_{tt})dt\\&=\frac{K}{2\theta_f}\|\nabla p(0)\|^2_{L^2(\Omega)}+\big(\phi,p(s)-p(0)\big)+\langle\phi_1,p(s)-p(0)\rangle.\end{aligned}\tag{2.53}$$

Differentiating (2.18) one time with respect to t and setting $\mathbf{v}=\boldsymbol{\tau}_{tt}$, we get

$$\frac{\gamma}{2}\frac{d}{dt}\|\varepsilon(\boldsymbol{\tau}_t)\|^2_{L^2(\Omega)}-(\delta_t,\mathrm{div}\boldsymbol{\tau}_{tt})=0.\tag{2.54}$$

Using the equality $q_t=\chi_1\varpi_t-\chi_3\delta_t-\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_{tt}$ and (2.54), we have

$$\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*\chi_1(\varpi_t,\mathrm{div}\boldsymbol{\tau}_{tt})$$

$$
\begin{aligned}
&=\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*\big(\mathrm{div}\boldsymbol{\tau}_{tt},\chi_3\delta_t+\mathrm{div}\boldsymbol{\tau}_t+\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_{tt}\big)\\
&=2\chi_3\lambda^*(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_t,\mathrm{div}\boldsymbol{\tau}_t)+(\lambda^*)^2\chi_3(\mathrm{div}\boldsymbol{\tau}_t,\mathrm{div}\boldsymbol{\tau}_{tt})\\
&=\gamma\lambda^*\chi_3\frac{d}{dt}\|\varepsilon(\boldsymbol{\tau}_t)\|^2_{L^2(\Omega)}+\lambda^*\|\mathrm{div}\boldsymbol{\tau}_t\|^2_{L^2(\Omega)}+\frac{(\lambda^*)^2\chi_3}{2}\frac{d}{dt}\|\mathrm{div}\boldsymbol{\tau}_t\|^2_{L^2(\Omega)}. \qquad (2.55)
\end{aligned}
$$

Using (2.55) and (2.53), we obtain

$$
\begin{aligned}
&\int_0^s\gamma\|\varepsilon(\boldsymbol{\tau}_t)\|^2_{L^2(\Omega)}+\chi_2\|\varpi_t\|^2_{L^2(\Omega)}+\chi_3\|\delta_t\|^2_{L^2(\Omega)}+\lambda^*\|\mathrm{div}\boldsymbol{\tau}_t\|^2_{L^2(\Omega)}dt\\
&\quad+\gamma\lambda^*\chi_3\big\|\varepsilon\big(\boldsymbol{\tau}_t(s)\big)\big\|^2_{L^2(\Omega)}+\frac{(\lambda^*)^2\chi_3}{2}\|\mathrm{div}\boldsymbol{\tau}_t(s)\|^2_{L^2(\Omega)}+\frac{K}{2\theta_f}\|\nabla p(s)\|^2_{L^2(\Omega)}\\
=&\gamma\lambda^*\chi_3\big\|\varepsilon\big(\boldsymbol{\tau}_t(0)\big)\big\|^2_{L^2(\Omega)}+\frac{(\lambda^*)^2\chi_3}{2}\|\mathrm{div}\boldsymbol{\tau}_t(0)\|^2_{L^2(\Omega)}+\frac{K}{2\theta_f}\|\nabla p(0)\|^2_{L^2(\Omega)}\\
&+\big(\phi,p(s)-p(0)\big)+\langle\phi_1,p(s)-p(0)\rangle, \qquad (2.56)
\end{aligned}
$$

which implies that (2.47) holds.

Differentiating (2.19) twice with respect to t and setting $\varphi=\delta_t$, and differentiating (2.20) one time with respect to t and setting $\psi=p_t=\chi_1\delta_t+\chi_2\varpi_t$ in (2.20), we get

$$
\frac{\chi_3}{2}\frac{d}{dt}\|\delta_t\|^2_{L^2(\Omega)}+(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{ttt},\delta_t)=\chi_1(\varpi_{tt},\delta_t), \qquad (2.57)
$$

$$
\chi_1(\varpi_{tt},\delta_t)+\lambda^*\chi_1(\varpi_{tt},\mathrm{div}\boldsymbol{\tau}_{tt})+\frac{\chi_2}{2}\frac{d}{dt}\|\varpi_t\|^2_{L^2(\Omega)}+\frac{K}{\theta_f}\|\nabla p_t\|^2_{L^2(\Omega)}=0. \qquad (2.58)
$$

Adding (2.54), (2.57) and (2.58), and integrating in t from 0 to $s\in[0,T]$, we have

$$
\begin{aligned}
&\frac{\gamma}{2}\|\varepsilon(\boldsymbol{\tau}_t)(s)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi_t(s)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\delta_t(s)\|^2_{L^2(\Omega)}+\frac{K}{\theta_f}\int_0^s\|\nabla p_t\|^2_{L^2(\Omega)}dt\\
&\quad+\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{ttt},\delta_t)+\lambda^*\chi_1(\varpi_{tt},\mathrm{div}\boldsymbol{\tau}_{tt})\\
=&\frac{\gamma}{2}\|\varepsilon(\boldsymbol{\tau}_t)(0)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi_t(0)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\delta_t(0)\|^2_{L^2(\Omega)}. \qquad (2.59)
\end{aligned}
$$

Using the equality $q_{tt}=\chi_1\varpi_{tt}-\chi_3\delta_{tt}-\lambda^*\chi_3\mathrm{div}q_{ttt}$, we have

$$
\begin{aligned}
&\lambda^*\chi_3(\mathrm{div}\boldsymbol{\tau}_{ttt},\delta_t)+\lambda^*\chi_1(\varpi_{tt},\mathrm{div}\boldsymbol{\tau}_{tt})\\
=&\lambda^*\chi_3\left[\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)-(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_{tt})\right]+\lambda^*\chi_1(\varpi_{tt},\mathrm{div}\boldsymbol{\tau}_{tt})\\
=&\lambda^*\chi_3\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_{tt},\chi_1\varpi_{tt}-\chi_3\delta_{tt})\\
=&\lambda^*\chi_3\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_{tt},\mathrm{div}\boldsymbol{\tau}_{tt}+\lambda^*\chi_3\mathrm{div}\boldsymbol{\tau}_{ttt})\\
=&\lambda^*\chi_3\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\lambda^*(\mathrm{div}\boldsymbol{\tau}_{tt},\mathrm{div}\boldsymbol{\tau}_{tt})+\frac{(\lambda^*)^2\chi_3}{2}\frac{d}{dt}(\mathrm{div}\boldsymbol{\tau}_{tt},\mathrm{div}\boldsymbol{\tau}_{tt}). \qquad (2.60)
\end{aligned}
$$

It is easy to check that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\delta_t,\delta_t)+\frac{d}{dt}(\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt},\delta_t)+\frac{1}{2}\frac{d}{dt}(\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt},\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt})\\=&\frac{1}{2}\frac{d}{dt}(\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt}+\delta_t,\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt}+\delta_t).\end{aligned}\tag{2.61}$$

Using (2.61), (2.60) and (2.59), we get

$$\begin{aligned}&\int_0^s\lambda^*\|(\mathrm{div}\boldsymbol{\tau})_{tt}\|^2_{L^2(\Omega)}dt+\frac{\gamma}{2}\|\varepsilon(\boldsymbol{\tau}_t)(s)\|^2_{L^2(\Omega)}\\&\quad+\frac{\chi_2}{2}\|\varpi_t(s)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt}(s)+\delta_t(s)\|^2_{L^2(\Omega)}+\frac{K}{\theta_f}\int_0^s\|\nabla p_t\|^2_{L^2(\Omega)}dt\\=&\frac{\gamma}{2}\|\varepsilon(\boldsymbol{\tau}_t)(0)\|^2_{L^2(\Omega)}+\frac{\chi_2}{2}\|\varpi_t(0)\|^2_{L^2(\Omega)}+\frac{\chi_3}{2}\|\lambda^*\mathrm{div}\boldsymbol{\tau}_{tt}(0)+\delta_t(0)\|^2_{L^2(\Omega)},\end{aligned}\tag{2.62}$$

which implies that (2.48) holds. (2.49) follows immediately from the following inequality:

$$(\varpi_{tt},\psi)=-\frac{1}{\theta_f}(K\nabla p_t,\nabla\psi)\le\frac{K_2}{\theta_f}\|\nabla p_t\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)},$$

(2.48) and the definition of the $H^1(\Omega)'$-norm. The proof is complete. □

Theorem 2.2. *Let* $\boldsymbol{\tau}_0\in\mathbf{H}^1(\Omega),\mathbf{F}\in\mathbf{L}^2(\Omega),\mathbf{F}_1\in\mathbf{L}^2(\partial\Omega),p_0\in L^2(\Omega),\phi\in L^2(\Omega)$, *and* $\phi_1\in L^2(\partial\Omega)$. *Suppose* $a_0>0$ *and* $(\mathbf{F},\mathbf{v})+\langle\mathbf{F}_1,\mathbf{v}\rangle=0$ *for any* $\mathbf{v}\in\mathbf{RM}$. *Then there exists a unique weak solution to the problem* (2.5)-(2.10) *in the sense of Definition* 2.2. *Likewise, there exists a unique weak solution to the problem* (1.1)-(1.2) *with* (2.1)-(2.3) *in the sense of Definition* 2.1.

Proof. The existence of weak solution can be easily proved by using the standard Galerkin method and the compactness argument (cf. [29]). Lemmas 2.1 and 2.2 provide the required uniform estimates for the Galerkin approximate solutions, since the derivation is standard, here we omit the details. Next, we prove the uniqueness of the weak solution of the problem (2.5)-(2.10). Lemma 2.4 and Theorem 2.1 give the priori estimates for the weak solution. Since $C_\varpi(t)=(\varpi(\cdot,t),1)=(\varpi_0,1)+[(\phi,1)+\langle\phi_1,1\rangle]$. It is easy to check that ϖ is unique. We assume that $(\boldsymbol{\tau}_1,\delta_1,\varpi_1)$ and $(\boldsymbol{\tau}_2,\delta_2,\varpi_2)$ are the different solutions of the problem (2.18)-(2.22). Using (2.18) and (2.19), we obtain

$$\gamma\big(\varepsilon(\boldsymbol{\tau}_1)-\varepsilon(\boldsymbol{\tau}_2),\varepsilon(\mathbf{v})\big)-(\delta_1-\delta_2,\mathrm{div}\mathbf{v})=0,\qquad\forall\mathbf{v}\in\mathbf{H}^1(\Omega),\tag{2.63}$$

$$\chi_3(\delta_1-\delta_2,\varphi)+(\mathrm{div}\boldsymbol{\tau}_1-\mathrm{div}\boldsymbol{\tau}_2,\varphi)+\lambda^*\chi_3\big((\mathrm{div}\boldsymbol{\tau}_1-\mathrm{div}\boldsymbol{\tau}_2)_t,\varphi\big)=0,\quad\forall\varphi\in L^2(\Omega).\tag{2.64}$$

Adding (2.63) and (2.64), letting $\mathbf{v}=\boldsymbol{\tau}_1-\boldsymbol{\tau}_2,\varphi=\delta_1-\delta_2$, we have

$$\frac{\lambda^*\chi_3\gamma}{2}\frac{d}{dt}\|\varepsilon(\boldsymbol{\tau}_1)-\varepsilon(\boldsymbol{\tau}_2)\|^2_{L^2(\Omega)}+\gamma\|\varepsilon(\boldsymbol{\tau}_1)-\varepsilon(\boldsymbol{\tau}_2)\|^2_{L^2(\Omega)}+\chi_3\|\delta_1-\delta_2\|^2_{L^2(\Omega)}=0.\tag{2.65}$$

Using (2.65) and the initial value $\boldsymbol{\tau}_0$, we obtain

$$\boldsymbol{\tau}_1=\boldsymbol{\tau}_2,\quad\delta_1=\delta_2.$$

Since $p=\chi_1\delta+\chi_2\varpi, q=\chi_1\varpi-\chi_3\delta$, so we have

$$p_1=p_2, \quad q_1=q_2.$$

Hence, the solution of the problem (2.18)-(2.22) is unique.

Taking the above similar argument, we can prove the uniqueness of weak solution of the problem (1.1)-(1.2) with (2.1)-(2.3), here we omit the details of the proof. The proof is complete. □

3 Fully discrete multiphysics finite element methods

3.1 Formulation of fully discrete multiphysics finite element methods

Let $\mathcal{T}_h$ be a quasi-uniform triangulation or rectangular partition of Ω with maximum mesh size h, and $\bar{\Omega}=\bigcup_{\mathcal{K}\in\mathcal{T}_h}\bar{\mathcal{K}}$. The time interval $[0,T]$ is divided into N equal intervals, denoted by $[t_{n-1},t_n], n=1,2,\ldots,N$, and $\Delta t=T/N$, then $t_n=n\Delta t$. In this work, we use backward Euler method and denote $d_t v^n := (v^n-v^{n-1})/\Delta t$.

A number of mixed finite element spaces $(\mathbf{X}_h, M_h)$ can be chosen arbitrarily (see Remark 3.1). For the convenience of numerical analysis, one can choose some stable mixed finite element space pairs (cf. [5]), that is, $\mathbf{X}_h\subset\mathbf{H}^1(\Omega)$ and $M_h\subset L^2(\Omega)$ satisfy the inf-sup condition

$$\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{(\mathrm{div}\mathbf{v}_h,\varphi_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)}}\geq\beta_0\|\varphi_h\|_{L^2(\Omega)}, \quad \forall\varphi_h\in M_{0h}:=M_h\cap L_0^2(\Omega), \quad \beta_0>0. \tag{3.1}$$

A well-known example is the following Taylor-Hood element (cf. [1,5]):

$$\begin{aligned}\mathbf{X}_h&=\left\{\mathbf{v}_h\in\mathbf{C}^0(\overline{\Omega}):\ \mathbf{v}_h|_{\mathcal{K}}\in\mathbf{P}_2(\mathcal{K}),\ \forall\mathcal{K}\in\mathcal{T}_h\right\},\\ M_h&=\left\{\varphi_h\in C^0(\overline{\Omega}):\ \varphi_h|_{\mathcal{K}}\in P_1(\mathcal{K}),\ \forall\mathcal{K}\in\mathcal{T}_h\right\}.\end{aligned}$$

Finite element approximation space W_h for ϖ variable can be chosen independently, any piecewise polynomial space is acceptable provided that $W_h\supset M_h$, the most convenient choice is $W_h=M_h$.

Define

$$\mathbf{V}_h:=\left\{\mathbf{v}_h\in\mathbf{X}_h:\ (\mathbf{v}_h,\mathbf{r})=0,\ \forall\mathbf{r}\in\mathbf{RM}\right\}, \tag{3.2}$$

it is easy to check that $\mathbf{X}_h=\mathbf{V}_h\bigoplus\mathbf{RM}$. It was proved in [12] that there holds the following inf-sup condition:

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(\mathrm{div}\mathbf{v}_h,\varphi_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)}}\geq\beta_1\|\varphi_h\|_{L^2(\Omega)}, \quad \forall\varphi_h\in M_{0h}, \quad \beta_1>0. \tag{3.3}$$

We also recall the following inverse inequality for polynomial functions [4–6]:

$$\|\nabla\varphi_h\|_{L^2(\mathcal{K})}\leq c_1h^{-1}\|\varphi_h\|_{L^2(\mathcal{K})}, \quad \forall\varphi_h\in P_r(\mathcal{K}), \quad \mathcal{K}\in T_h. \tag{3.4}$$

Now, we propose the fully discrete multiphysics finite element algorithm for the problem (2.5)-(2.7).

Algorithm 1: Multiphysics Finite Element Algorithm (MFEA).

1: Compute $\boldsymbol{\tau}_h^0 \in \mathbf{V}_h$ and $q_h^0 \in W_h$ by $\boldsymbol{\tau}_h^0 = \boldsymbol{\tau}_0, p_h^0 = p_0$.

2: For $n=0,1,2,\ldots$, do the following two steps.

Step 1: Solve for $(\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1},\varpi_h^{n+1}) \in \mathbf{V}_h \times M_h \times W_h$ such that

$$\begin{aligned}
&\lambda^*\big(\mathrm{div}\boldsymbol{\tau}_h^{n+1},\mathrm{div}\mathbf{v}_h\big)+\gamma\big(\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big),\varepsilon(\mathbf{v}_h)\big)-\big(\delta_h^{n+1},\mathrm{div}\mathbf{v}_h\big)\\
&=(\mathbf{F},\mathbf{v}_h)+\langle \mathbf{F}_1,\mathbf{v}_h\rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \qquad (3.5)\\
&\chi_3\big(\delta_h^{n+1},\varphi_h\big)+\big(\mathrm{div}\boldsymbol{\tau}_h^{n+1},\varphi_h\big)+\lambda^*\chi_3\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\varphi_h\big)\\
&=\chi_1\big(\varpi_h^{n+\theta},\varphi_h\big), \qquad \forall \varphi_h \in M_h, \qquad (3.6)\\
&\big(d_t\varpi_h^{n+1},\psi_h\big)+\frac{1}{\theta_f}\Big(K\big(\nabla\big(\chi_1\delta_h^{n+1}+\chi_2\varpi_h^{n+1}+\lambda^*\chi_1 d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)-\rho_f\mathbf{g}\big),\nabla\psi_h\Big)\\
&=(\phi,\psi_h)+\langle\phi_1,\psi_h\rangle, \quad \forall \psi_h \in W_h, \qquad (3.7)
\end{aligned}$$

where $\theta=0,1$.

Step 2: Update p_h^{n+1} and q_h^{n+1} by

$$p_h^{n+1}=\chi_1\delta_h^{n+1}+\chi_2\varpi_h^{n+\theta}+\lambda^*\chi_1 d_t\mathrm{div}_h^{n+1}, \qquad (3.8)$$

$$q_h^{n+1}=\chi_1\varpi_h^{n+\theta}-\chi_3\delta_h^{n+1}-\lambda^*\chi_3 d_t\mathrm{div}_h^{n+1}. \qquad (3.9)$$

Remark 3.1. When $\theta=0$, the problem (3.5)-(3.7) is a decoupled problem (a generalized Stokes problem and a diffusion problem). And the generalized Stokes problem (3.5)-(3.6) is not a saddle point problem for the finite constant β so that the finite element spaces $(\mathbf{X}_h,M_h)$ can be chosen arbitrarily. However, there exists the mesh constraint $\Delta t=\mathcal{O}(h^2)$ in the numerical analysis. When $\theta=1$, the problem (3.5)-(3.7) is a coupled problem, and there does not exist the mesh constraint.

Lemma 3.1. *Let* $\{(\boldsymbol{\tau}_h^n,\delta_h^n,\varpi_h^n)\}_{n\geq 0}$ *be defined by the (MFEA), then there hold*

$$\begin{aligned}
&(\varpi_h^n,1)=C_\varpi(t_n) && \textit{for} \quad n=0,1,2,\ldots, && (3.10)\\
&(\delta_h^n,1)=C_\delta(t_{n-1+\theta})=\frac{1}{(d\lambda^*\chi_3)/\Delta t+\chi_3\gamma+d}\\
&\qquad\times\left[\frac{d\lambda^*\chi_3}{\Delta t}(\delta_h^{n-1},1)+\chi_1\gamma C_\varpi(t_{n-1+\theta})-(\mathbf{F},\mathbf{x})-\langle\mathbf{F}_1,\mathbf{x}\rangle\right] && \textit{for} \quad n=1-\theta,1,2,\ldots, && (3.11)\\
&\big\langle\boldsymbol{\tau}_h^n\cdot\mathbf{n},1\big\rangle=C_{\boldsymbol{\tau}}(t_{n-1+\theta}) && \textit{for} \quad n=1-\theta,1,2,\ldots. && (3.12)
\end{aligned}$$

Proof. Taking $\psi_h=1$ in (3.7), we have

$$\big(d_t\varpi_h^{n+1},1\big)=(\phi,1)+\langle\phi_1,1\rangle. \qquad (3.13)$$

Summing (3.13) over n from 0 to ℓ (≥ 0), we get

$$\left(\varpi_h^{\ell+1},1\right)=(\varpi_0,1)+[(\phi,1)+\langle\phi_1,1\rangle]t_{\ell+1}=C_\varpi(t_{\ell+1}),\quad \ell=0,1,2,\ldots, \tag{3.14}$$

which implies that (3.10) holds.

Taking $\mathbf{v}_h=\mathbf{x}$ in (3.5) and $\varphi_h=1$ in (3.6), we get

$$\gamma\left(\varepsilon(\boldsymbol{\tau}_h^{n+1}),\mathbf{I}\right)-d\left(\delta_h^{n+1},1\right)=(\mathbf{F},\mathbf{x})+\langle\mathbf{F}_1,\mathbf{x}\rangle, \tag{3.15}$$

$$\chi_3\left(\delta_h^{n+1},1\right)+\left(\mathrm{div}\boldsymbol{\tau}_h^{n+1},1\right)+\lambda^*\chi_3\left(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},1\right)=\chi_1C_\varpi(t_{n+\theta}). \tag{3.16}$$

Substituting (3.15) into (3.16), we have

$$d\lambda^*\chi_3\left(d_t\delta_h^{n+1},1\right)+(\chi_3\gamma+d)\left(\delta_h^{n+1},1\right)=\chi_1\gamma C_\varpi(t_{n+\theta})-(\mathbf{F},\mathbf{x})-\langle\mathbf{F}_1,\mathbf{x}\rangle. \tag{3.17}$$

Using (3.17), we get

$$\left(\frac{d\lambda^*\chi_3}{\Delta t}+\chi_3\gamma+d\right)\left(\delta_h^{n+1},1\right)=\frac{d\lambda^*\chi_3}{\Delta t}\left(\delta_h^n,1\right)+\chi_1\gamma C_\varpi(t_{n+\theta})-(\mathbf{F},\mathbf{x})-\langle\mathbf{F}_1,\mathbf{x}\rangle,$$

which implies (3.11) holds for all $n\geq 1-\theta$.

Using (3.10), (3.11), (3.16) and Gauss divergence theorem, we deduce that (3.12) holds. The proof is complete. □

Lemma 3.2. *Let* $\{(\boldsymbol{\tau}_h^n,\delta_h^n,\varpi_h^n)\}_{n\geq 0}$ *be defined by the (MFEA), then there holds the following inequality:*

$$J_{h,\theta}^{l+1}+S_{h,\theta}^{l+1}=J_{h,\theta}^0\quad for\quad l\geq 0,\quad \theta=0,1, \tag{3.18}$$

where

$$\begin{aligned}J_{h,\theta}^{l+1}=\frac{1}{2}\Big[&\gamma\left\|\varepsilon(\boldsymbol{\tau}_h^{l+1})\right\|_{L^2(\Omega)}^2+\chi_2\left\|\varpi_h^{l+\theta}\right\|_{L^2(\Omega)}^2+\chi_3\left\|\lambda^*d_t\mathrm{div}\boldsymbol{\tau}_h^{l+1}+\delta_h^{l+1}\right\|_{L^2(\Omega)}^2\\&+\frac{\gamma\lambda^*\chi_3\Delta t}{2}\left\|d_t\varepsilon(\boldsymbol{\tau}_h^{l+1})\right\|_{L^2(\Omega)}^2-2\left(\mathbf{F},\boldsymbol{\tau}_h^{l+1}\right)-2\left\langle\mathbf{F}_1,\boldsymbol{\tau}_h^{l+1}\right\rangle\Big],\\
S_{h,\theta}^{l+1}=\Delta t\sum_{n=0}^{l}\Big[&\lambda^*\left\|d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\right\|_{L^2(\Omega)}^2+\frac{\gamma\Delta t}{2}\left\|d_t\varepsilon(\boldsymbol{\tau}_h^{n+1})\right\|_{L^2(\Omega)}^2\\
&+\frac{1}{\theta_f}\left(K\nabla p_h^{n+1}-K\rho_f g,\nabla p_h^{n+1}\right)+\frac{\chi_2\Delta t}{2}\left\|d_t\varpi_h^{n+\theta}\right\|_{L^2(\Omega)}^2\\
&+\frac{\chi_3\Delta t}{2}\left\|d_t\delta_h^{n+1}\right\|_{L^2(\Omega)}^2+\frac{\chi_3\Delta t}{2}\left\|\lambda^*d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1}\right\|_{L^2(\Omega)}^2\\
&+\frac{\gamma\lambda^*\chi_3(\Delta t)^2}{2}\left\|d_t^2\varepsilon(\boldsymbol{\tau}_h^{n+1})\right\|_{L^2(\Omega)}^2-\left(\phi,p_h^{n+1}\right)-\left\langle\phi_1,p_h^{n+1}\right\rangle\\
&-(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\left(Kd_t\nabla\delta_h^{n+1},\nabla p_h^{n+1}\right)\\
&-(1-\theta)\frac{\chi_1\lambda^*\Delta t}{\theta_f}\left(Kd_t^2\nabla\mathrm{div}\boldsymbol{\tau}_h^{n+1},\nabla p_h^{n+1}\right)\Big].\end{aligned}$$

Proof. As for the case $\theta=1$, we can prove (3.18) by taking the same argument as the continuous case (PDE analysis, see Lemma 2.1), here we omit the details to save spaces. Next, we prove the case of $\theta=0$. Using (3.5), we can define ϖ_h^{-1} by

$$\chi_1\big(\varpi_h^{-1},\varphi_h\big)=\chi_3\big(\delta_h^0,\varphi_h\big)+\big(\mathrm{div}\boldsymbol{\tau}_h^0,\varphi_h\big)+\lambda^*\chi_3\big(d_t\mathrm{div}\boldsymbol{\tau}_h^0,\varphi_h\big). \tag{3.19}$$

Setting $\mathbf{v}_h=d_t\boldsymbol{\tau}_h^{n+1}$ in (3.5), $\varphi_h=\delta_h^{n+1}$ in (3.6) and $\psi_h=p_h^{n+1}$ in (3.7) after lowing the super-index from $n+1$ to n on both sides of (3.7), we get

$$\gamma\big(\varepsilon(\boldsymbol{\tau}_h^{n+1}),\varepsilon(d_t\boldsymbol{\tau}_h^{n+1})\big)-\big(\delta_h^{n+1},d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)=\big(\mathbf{F},d_t\boldsymbol{\tau}_h^{n+1}\big)+\big\langle\mathbf{F}_1,d_t\boldsymbol{\tau}_h^{n+1}\big\rangle, \tag{3.20}$$

$$\chi_3\big(d_t\delta_h^{n+1},\delta_h^{n+1}\big)+\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)+\lambda^*\chi_3\big(d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)=\chi_1\big(d_t\varpi_h^n,\delta_h^{n+1}\big), \tag{3.21}$$

$$\begin{aligned}&\big(d_t\varpi_h^n,p_h^{n+1}\big)+\frac{1}{\theta_f}\Big(K\big(\nabla\big(\chi_1\delta_h^n+\chi_2\varpi_h^n+\lambda^*\chi_1d_t\mathrm{div}\boldsymbol{\tau}_h^n\big)-\rho_f\mathbf{g}\big),\nabla p_h^{n+1}\Big)\\&=\big(\phi,p_h^{n+1}\big)+\big\langle\phi_1,p_h^{n+1}\big\rangle.\end{aligned} \tag{3.22}$$

Adding (3.20)-(3.21), we have

$$\begin{aligned}&\gamma\big(\varepsilon(\boldsymbol{\tau}_h^{n+1}),\varepsilon(d_t\boldsymbol{\tau}_h^{n+1})\big)+\chi_3\big(d_t\delta_h^{n+1},\delta_h^{n+1}\big)+\lambda^*\chi_3\big(d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)\\&\quad+\chi_1\big(d_t\varpi_h^n,\varpi_h^n\big)+\big(d_t\varpi_h^n,\lambda^*\chi_1d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)+\frac{1}{\theta_f}\big(K\nabla p_h^{n+1}-K\rho_fg,\nabla p_h^{n+1}\big)\\&\quad-\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla\delta_h^{n+1},\nabla p_h^{n+1}\big)-\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t^2\nabla\mathrm{div}\boldsymbol{\tau}_h^{n+1},\nabla p_h^{n+1}\big)\\&=\big(\mathbf{F},d_t\boldsymbol{\tau}_h^{n+1}\big)+\big\langle\mathbf{F}_1,d_t\boldsymbol{\tau}_h^{n+1}\big\rangle+\big(\phi,p_h^{n+1}\big)+\big\langle\phi_1,p_h^{n+1}\big\rangle.\end{aligned} \tag{3.23}$$

Using the equality

$$d_tq_h^{n+1}=\chi_1d_t\varpi_h^n-d_t\chi_3\delta_h^{n+1}-\lambda^*d_t^2\chi_3\mathrm{div}_h^{n+1}$$

and (3.23), we have

$$\begin{aligned}&\lambda^*\chi_3\big(d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)+\big(d_t\varpi_h^n,\lambda^*\chi_1d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)\\&=\lambda^*\chi_3d_t\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)-\lambda^*\chi_3\big(d_t\mathrm{div}\boldsymbol{\tau}_h^n,d_t\delta_h^{n+1}\big)+\big(d_t\varpi_h^n,\lambda^*\chi_1d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)\\&=\lambda^*\chi_3d_t\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)-\lambda^*\chi_3\big(d_t\mathrm{div}\boldsymbol{\tau}_h^n,d_t\delta_h^{n+1}\big)+\lambda^*\chi_3\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\delta_h^{n+1}\big)\\&\quad-\lambda^*\chi_3\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\delta_h^{n+1}\big)+\big(d_t\varpi_h^n,\lambda^*\chi_1d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)\\&=\lambda^*\chi_3d_t\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)+\lambda^*\chi_3\Delta t\big(d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\delta_h^{n+1}\big)\\&\quad+\lambda^*\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\chi_1d_t\varpi_h^n-\chi_3d_t\delta_h^{n+1}\big)\\&=\lambda^*\chi_3d_t\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)+\lambda^*\chi_3\Delta t\big(d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\delta_h^{n+1}\big)\\&\quad+\lambda^*\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}+\lambda^*\chi_3d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)\\&=\lambda^*\chi_3d_t\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\delta_h^{n+1}\big)+\lambda^*\chi_3\Delta t\big(d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\delta_h^{n+1}\big)\\&\quad+\lambda^*\big(d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)+\chi_3\big(\lambda^*d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1},\lambda^*d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big).\end{aligned} \tag{3.24}$$

Taking $\mathbf{v}_h=d_t^2\boldsymbol{\tau}_h^{n+1}$ in (3.5), we get

$$\gamma\Delta t\big(\varepsilon\big(d_t\boldsymbol{\tau}_h^{n+1}\big),\varepsilon\big(d_t^2\boldsymbol{\tau}_h^{n+1}\big)\big)-\Delta t\big(d_t\delta_h^{n+1},d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big)=0. \tag{3.25}$$

It is easy to check that

$$\big(d_t a_h^{n+1},a_h^{n+1}\big)=\frac{\Delta t}{2}\big\|d_t a_h^{n+1}\big\|_{L^2(\Omega)}^2+\frac{1}{2}d_t\big\|a_h^{n+1}\big\|_{L^2(\Omega)}^2, \tag{3.26}$$

where a can be chosen by $=\varepsilon(\boldsymbol{\tau}),\delta,\varpi$ and $\mathrm{div}\boldsymbol{\tau}$.

Using (3.26), (3.24), (3.25) and (3.23), we get

$$\begin{aligned}
&\frac{\gamma}{2}d_t\big\|\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2+\frac{\gamma\Delta t}{2}\big\|d_t\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2+\frac{\chi_3\Delta t}{2}\big\|d_t\delta_h^{n+1}\big\|_{L^2(\Omega)}^2\\
&\quad+\frac{\chi_3\Delta t}{2}\big\|\lambda^* d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big\|_{L^2(\Omega)}^2+\frac{\chi_3}{2}d_t\big\|\lambda^* d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}+\delta_h^{n+1}\big\|_{L^2(\Omega)}^2\\
&\quad+\frac{\chi_2\Delta t}{2}\big\|d_t\varpi_h^n\big\|_{L^2(\Omega)}^2+\frac{\chi_2}{2}d_t\big\|\varpi_h^n\big\|_{L^2(\Omega)}^2+\lambda^*\big\|d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big\|_{L^2(\Omega)}^2\\
&\quad+\frac{\gamma\lambda^*\chi_3\Delta t}{2}d_t\big\|d_t\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2+\frac{\gamma\lambda^*\chi_3(\Delta t)^2}{2}\big\|d_t^2\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2\\
&\quad+\frac{1}{\theta_f}\big(K\nabla p_h^{n+1}-K\rho_f\mathbf{g},\nabla p_h^{n+1}\big)-\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla\delta_h^{n+1},\nabla p_h^{n+1}\big)\\
&\quad-\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t^2\nabla\mathrm{div}\boldsymbol{\tau}_h^{n+1},\nabla p_h^{n+1}\big)\\
&=\big(\mathbf{F},d_t\boldsymbol{\tau}_h^{n+1}\big)+\big\langle\mathbf{F}_1,d_t\boldsymbol{\tau}_h^{n+1}\big\rangle+\big(\phi,p_h^{n+1}\big)+\big\langle\phi_1,p_h^{n+1}\big\rangle.
\end{aligned} \tag{3.27}$$

Applying the summation operator $\Delta t\sum_{n=0}^{l}$ to the both sides of (3.27), we see that (3.18) holds. The proof is complete. □

Lemma 3.3. *Let $\{(\boldsymbol{\tau}_h^n,\delta_h^n,\varpi_h^n)\}_{n\geq 0}$ be defined by the (MFEA) with $\theta=0$, then there holds the following inequality:*

$$J_{h,0}^{l+1}+\hat{S}_{h,0}^{l+1}\leq J_{h,0}^{0}\quad for\quad l\geq 0, \tag{3.28}$$

provided that $\Delta t=\mathcal{O}(h^2)$. Here

$$\begin{aligned}
\hat{S}_{h,0}^{l+1}=\Delta t\sum_{n=0}^{l}\Bigg[&\lambda^*\big\|d_t\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big\|_{L^2(\Omega)}^2+\frac{\gamma\Delta t}{4}\big\|d_t\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2\\
&+\frac{1}{\theta_f}\big(K\nabla p_h^{n+1}-K\rho_f\mathbf{g},\nabla p_h^{n+1}\big)+\frac{\chi_2\Delta t}{2}\big\|d_t\varpi_h^n\big\|_{L^2(\Omega)}^2\\
&+\frac{\chi_3\Delta t}{2}\big\|d_t\delta_h^{n+1}\big\|_{L^2(\Omega)}^2+\frac{\chi_3\Delta t}{4}\big\|\lambda^* d_t^2\mathrm{div}\boldsymbol{\tau}_h^{n+1}\big\|_{L^2(\Omega)}^2\\
&+\frac{\gamma\lambda^*\chi_3}{2}\big\|d_t^2\varepsilon\big(\boldsymbol{\tau}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2-\big(\phi,p_h^{n+1}\big)-\big\langle\phi_1,p_h^{n+1}\big\rangle\Bigg].
\end{aligned}$$

Proof. Using Cauchy-Schwarz inequality and (3.4), we get

$$\begin{aligned}
&\frac{\chi_1 \Delta t}{\theta_f}\left(K d_t \nabla \delta_h^{n+1}, \nabla p_h^{n+1}\right) \\
\leq &\frac{K_2^2 \chi_1^2}{K_1 \theta_f}\left\|\nabla \delta_h^{n+1}-\nabla \delta_h^n\right\|_{L^2(\Omega)}^2+\frac{K_1}{4 \theta_f}\left\|\nabla p_h^{n+1}\right\|_{L^2(\Omega)}^2 \\
\leq &\frac{K_2^2 \chi_1^2}{K_1 \theta_f h^2}\left\|\delta_h^{n+1}-\delta_h^n\right\|_{L^2(\Omega)}^2+\frac{K_1}{4 \theta_f}\left\|\nabla p_h^{n+1}\right\|_{L^2(\Omega)}^2, \qquad (3.29)\\
&\frac{\chi_1 \lambda^* \Delta t}{\theta_f}\left(K \nabla d_t^2 \operatorname{div} \boldsymbol{\tau}_h^{n+1}, \nabla p_h^{n+1}\right) \\
\leq &\frac{\chi_1 K_2 \Delta t}{\theta_f}\left\|\lambda^* \nabla d_t^2 \operatorname{div} \boldsymbol{\tau}_h^{n+1}\right\|_{L^2(\Omega)}\left\|\nabla p_h^{n+1}\right\|_{L^2(\Omega)} \\
\leq &\frac{K_2^2 \chi_1^2(\Delta t)^2}{K_1 \theta_f h^2}\left\|\lambda^* d_t^2 \operatorname{div} \boldsymbol{\tau}_h^{n+1}\right\|_{L^2(\Omega)}^2+\frac{K_1}{4 \theta_f}\left\|\nabla p_h^{n+1}\right\|_{L^2(\Omega)}^2. \qquad (3.30)
\end{aligned}$$

To bound the first term on the right-hand side of (3.29), using (3.3), we have

$$\begin{aligned}
\left\|\delta_h^{n+1}-\delta_h^n\right\|_{L^2(\Omega)} &\leq \frac{1}{\beta_1} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\left(\operatorname{div} \mathbf{v}_h, \delta_h^{n+1}-\delta_h^n\right)}{\left\|\nabla \mathbf{v}_h\right\|_{L^2(\Omega)}} \\
&\leq \frac{1}{\beta_1} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\gamma\left(\varepsilon\left(\boldsymbol{\tau}_h^{n+1}-\boldsymbol{\tau}_h^n\right), \varepsilon\left(\mathbf{v}_h\right)\right)}{\left\|\nabla \mathbf{v}_h\right\|_{L^2(\Omega)}} \leq \frac{\gamma \Delta t}{\beta_1}\left\|d_t \varepsilon\left(\boldsymbol{\tau}_h^{n+1}\right)\right\|_{L^2(\Omega)}. \qquad (3.31)
\end{aligned}$$

Substituting (3.31) into (3.29) and using (3.18), we deduce that (3.28) holds if

$$\Delta t \leq \min \left\{\frac{K_1 \theta_f \beta_1^2 h^2}{4 \gamma \chi_1^2 K_2^2}, \frac{K_1 \theta_f \chi_3 h^2}{4 \chi_1^2 K_2^2}\right\}.$$

The proof is complete. □

3.2 Error estimates

To derive the optimal order error estimates of the fully discrete multiphysics finite element method, for any $\varphi \in L^2(\Omega)$, we firstly define $L^2(\Omega)$-projection operators $\mathcal{Q}_h: L^2(\Omega) \rightarrow X_h^k$ by

$$\left(\mathcal{Q}_h \varphi, \psi_h\right)=\left(\varphi, \psi_h\right), \quad \psi_h \in X_h^k, \qquad (3.32)$$

where $X_h^k:=\left\{\psi_h \in C^0: \psi_h|_E \in P_k(E), \forall E \in \mathcal{T}_h\right\}$, k is the degree of piecewise polynomial on $\mathcal{K}$.

Next, for any $\varphi \in H^1(\Omega)$, we define its elliptic projection $\mathcal{S}_h: H^1(\Omega) \rightarrow X_h^k$ by

$$\begin{aligned}
&\left(K \nabla \mathcal{S}_h \varphi, \nabla \varphi_h\right)=\left(K \nabla \varphi, \nabla \varphi_h\right), \quad \forall \varphi_h \in X_h^k, \qquad (3.33)\\
&\left(\mathcal{S}_h \varphi, 1\right)=(\varphi, 1). \qquad (3.34)
\end{aligned}$$

Finally, for any $\mathbf{v}\in\mathbf{H}^1(\Omega)$, we define its elliptic projection $\mathcal{R}_h:\mathbf{H}^1(\Omega)\to\mathbf{V}_h^k$ by

$$\big(\varepsilon(\mathcal{R}_h\mathbf{v}),\varepsilon(\mathbf{w}_h)\big)=\big(\varepsilon(\mathbf{v}),\varepsilon(\mathbf{w}_h)\big),\quad \forall\mathbf{w}_h\in\mathbf{V}_h^k, \tag{3.35}$$

where

$$\mathbf{V}_h^k:=\big\{\mathbf{v}_h\in\mathbf{C}^0:\ \mathbf{v}_h|_{\mathcal{K}}\in\mathbf{P}_k(\mathcal{K}),(\mathbf{v}_h,\mathbf{r})=0,\forall\mathbf{r}\in\mathbf{RM}\big\},$$

k is the degree of the piecewise polynomial on $\mathcal{K}$. From [4], we know that $\mathcal{Q}_h,\mathcal{S}_h$ and $\mathcal{R}_h$ satisfy

$$\begin{aligned}
&\|\mathcal{Q}_h\varphi-\varphi\|_{L^2(\Omega)}+h\|\nabla(\mathcal{Q}_h\varphi-\varphi)\|_{L^2(\Omega)}\\
\le{}& Ch^{s+1}\|\varphi\|_{H^{s+1}(\Omega)},\quad \forall\varphi\in H^{s+1}(\Omega),\quad 0\le s\le k, &(3.36)\\
&\|\mathcal{S}_h\varphi-\varphi\|_{L^2(\Omega)}+h\|\nabla(\mathcal{S}_h\varphi-\varphi)\|_{L^2(\Omega)}\\
\le{}& Ch^{s+1}\|\varphi\|_{H^{s+1}(\Omega)},\quad \forall\varphi\in H^{s+1}(\Omega),\quad 0\le s\le k, &(3.37)\\
&\|\mathcal{R}_h\mathbf{v}-\mathbf{v}\|_{L^2(\Omega)}+h\|\nabla(\mathcal{R}_h\mathbf{v}-\mathbf{v})\|_{L^2(\Omega)}\\
\le{}& Ch^{s+1}\|\mathbf{v}\|_{H^{s+1}(\Omega)},\quad \forall\mathbf{v}\in\mathbf{H}^{s+1}(\Omega),\quad 0\le s\le k. &(3.38)
\end{aligned}$$

To derive the error estimates, we introduce the following notations:

$$\begin{aligned}
&E_{\boldsymbol{\tau}}^n=\boldsymbol{\tau}(t_n)-\boldsymbol{\tau}_h^n,\quad E_\delta^n=\delta(t_n)-\delta_h^n,\quad E_\varpi^n=\varpi(t_n)-\varpi_h^n,\\
&E_p^n=p(t_n)-p_h^n,\quad E_q^n=q(t_n)-q_h^n.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&E_p^n=\chi_1E_\delta^n+\chi_2E_\varpi^n+\lambda^*\chi_1d_t\mathrm{div}E_{\boldsymbol{\tau}}^n,\\
&E_q^n=\chi_1E_\varpi^n-\chi_3E_\delta^n-\lambda^*\chi_3d_t\mathrm{div}E_{\boldsymbol{\tau}}^n.
\end{aligned}\tag{3.39}$$

Also, we denote

$$\begin{aligned}
&E_{\boldsymbol{\tau}}^n=\boldsymbol{\tau}(t_n)-\mathcal{R}_h\big(\boldsymbol{\tau}(t_n)\big)+\mathcal{R}_h\big(\boldsymbol{\tau}(t_n)\big)-\boldsymbol{\tau}_h^n=:Y_{\boldsymbol{\tau}}^n+Z_{\boldsymbol{\tau}}^n,\\
&E_\delta^n=\delta(t_n)-\mathcal{S}_h\big(\delta(t_n)\big)+\mathcal{S}_h\big(\delta(t_n)\big)-\delta_h^n=:Y_\delta^n+Z_\delta^n,\\
&E_\varpi^n=\varpi(t_n)-\mathcal{S}_h\big(\varpi(t_n)\big)+\mathcal{S}_h\big(\varpi(t_n)\big)-\varpi_h^n=:Y_\varpi^n+Z_\varpi^n,\\
&E_p^n=p(t_n)-\mathcal{S}_h\big(p(t_n)\big)+\mathcal{S}_h\big(p(t_n)\big)-p_h^n=:Y_p^n+Z_p^n,\\
&E_\delta^n=\delta(t_n)-\mathcal{Q}_h\big(\delta(t_n)\big)+\mathcal{Q}_h\big(\delta(t_n)\big)-\delta_h^n=:F_\delta^n+G_\delta^n,\\
&E_\varpi^n=\varpi(t_n)-\mathcal{Q}_h\big(\varpi(t_n)\big)+\mathcal{Q}_h\big(\varpi(t_n)\big)-\varpi_h^n=:F_\varpi^n+G_\varpi^n,\\
&E_p^n=p(t_n)-\mathcal{Q}_h\big(p(t_n)\big)+\mathcal{Q}_h\big(p(t_n)\big)-p_h^n=:F_p^n+G_p^n.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&G_p^n=\chi_1G_\delta^n+\chi_2G_\varpi^n+\lambda^*\chi_1d_t\mathrm{div}Z_{\boldsymbol{\tau}}^n,\\
&Z_p^n=\chi_1Z_\delta^n+\chi_2Z_\varpi^n+\lambda^*\chi_1d_t\mathrm{div}Z_{\boldsymbol{\tau}}^n,\\
&G_q^n=Z_q^n=\mathrm{div}Z_{\boldsymbol{\tau}}^n=\chi_1G_\varpi^n-\chi_3G_\delta^n-\lambda^*\chi_3d_t\mathrm{div}Z_{\boldsymbol{\tau}}^n.
\end{aligned}$$

Lemma 3.4. *Let $\{(\boldsymbol{\tau}_h^n,\delta_h^n,\varpi_h^n)\}_{n\geq 0}$ be generated by the (MFEA) and $Y_{\boldsymbol{\tau}}^n,Z_{\boldsymbol{\tau}}^n,Y_\delta^n,Z_\delta^n,Y_\varpi^n$ and Z_ϖ^n be defined as above. Then there holds*

$$
\begin{aligned}
&\mathcal{E}_h^{l+1}+\Delta t\sum_{n=0}^{l}\Big[\frac{\Delta t}{2}\Big(\chi_3\big\|\lambda^* d_t^2\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1}\big\|^2_{L^2(\Omega)}+\gamma\big\|d_t\varepsilon\big(Z_{\boldsymbol{\tau}}^{n+1}\big)\big\|^2_{L^2(\Omega)}+\chi_3\big\|d_tG_\delta^{n+1}\big\|^2_{L^2(\Omega)}\\
&\qquad\qquad+\chi_2\big\|d_tG_\varpi^{n+\theta}\big\|^2_{L^2(\Omega)}+\gamma\chi_3\lambda^*\Delta t\big\|d_t^2\varepsilon\big(Z_{\boldsymbol{\tau}}^{n+1}\big)\big\|^2_{L^2(\Omega)}\Big)\\
&\qquad\qquad+\lambda^*\big\|d_t\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1}\big\|^2_{L^2(\Omega)}+\frac{1}{\theta_f}\big(K\nabla\hat{Z}_p^{n+1},\nabla\hat{Z}_p^{n+1}\big)\Big]\\
=&\mathcal{E}_h^0+\Delta t\sum_{n=0}^{l}\Big[\chi_3\lambda^*\Delta t\big(d_tF_\delta^{n+1},\mathrm{div}d_t^2Z_{\boldsymbol{\tau}}^{n+1}\big)+\big(F_\delta^{n+1},\mathrm{div}d_tZ_{\boldsymbol{\tau}}^{n+1}\big)\\
&\qquad\qquad+\lambda^*\chi_3\big(\mathbf{R}_h^{n+1},G_\delta^{n+1}\big)-\big(\mathrm{div}d_tY_{\boldsymbol{\tau}}^{n+1},G_\delta^{n+1}\big)\Big]\\
&+\Delta t\sum_{n=0}^{l}\Big[\big(R_h^{n+\theta},\hat{Z}_p^{n+1}\big)+\chi_1(1-\theta)\Delta t\big(d_t^2\varpi(t_{n+1}),G_\delta^{n+1}\big)\\
&\qquad\qquad+(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla Z_\delta^{n+1},\nabla\hat{Z}_p^{n+1}\big)\Big]\\
&+\Delta t\sum_{n=0}^{l}\big(d_tG_\varpi^{n+\theta},Y_p^{n+1}-F_p^{n+1}\big)\\
&+\Delta t\sum_{n=0}^{l}\Big[(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1},\nabla\hat{Z}_p^{n+1}\big)-\lambda^*\chi_3\big(d_t^2\mathrm{div}Y_{\boldsymbol{\tau}}^{n+1},G_\delta^{n+1}\big)\Big],
\end{aligned}\tag{3.40}
$$

where

$$
\begin{aligned}
&\hat{Z}_p^{n+1}=F_p^{n+1}-Y_p^{n+1}+\chi_1G_\delta^{n+1}+\chi_2G_\varpi^{n+\theta},\\
&\mathcal{E}_h^{l+1}=\frac{1}{2}\Big[\gamma\big\|\varepsilon\big(Z_{\boldsymbol{\tau}}^{l+1}\big)\big\|^2_{L^2(\Omega)}+\chi_2\big\|G_\varpi^{l+\theta}\big\|^2_{L^2(\Omega)}+\chi_3\big\|\lambda^*d_t\mathrm{div}Z_{\boldsymbol{\tau}}^{l+1}+G_\delta^{l+1}\big\|^2_{L^2(\Omega)}\\
&\qquad\qquad+\gamma\chi_3\lambda^*\Delta t\big\|d_t\varepsilon\big(Z_{\boldsymbol{\tau}}^{l+1}\big)\big\|^2_{L^2(\Omega)}\Big],\\
&R_h^{n+1}=-\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)\varpi_{tt}(s)ds,\\
&\mathbf{R}_h^{n+1}=-\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)\mathrm{div}\boldsymbol{\tau}_{tt}(s)ds.
\end{aligned}
$$

Proof. Subtracting (3.5) from (2.22), (3.6) from (2.32), (3.7) from (2.33), respectively, we get

$$
\gamma\big(\varepsilon\big(E_{\boldsymbol{\tau}}^{n+1}\big),\varepsilon(\mathbf{v}_h)\big)-\big(E_\delta^{n+1},\mathrm{div}\mathbf{v}_h\big)=0,\qquad\forall\mathbf{v}_h\in\mathbf{V}_h,\tag{3.41}
$$

$$
\begin{aligned}
&\chi_3\big(E_\delta^{n+1},\varphi_h\big)+\big(\mathrm{div}E_{\boldsymbol{\tau}}^{n+1},\varphi_h\big)+\lambda^*\chi_3\big(d_t\mathrm{div}E_{\boldsymbol{\tau}}^{n+1},\mathrm{div}\mathbf{v}_h\big)\\
=&\chi_1\big(E_\varpi^{n+\theta},\varphi_h\big)+\lambda^*\chi_3\big(\mathbf{R}_h^{n+1},\mathrm{div}\mathbf{v}_h\big)+\chi_1(1-\theta)\Delta t\big(d_t\varpi(t_{n+1}),\varphi_h\big),\quad\forall\varphi_h\in M_h,
\end{aligned}\tag{3.42}
$$

$$\begin{aligned}&\big(d_tE_\varpi^{n+\theta},\psi_h\big)+\frac{1}{\theta_f}\big(K\nabla E_p^{n+1},\nabla\psi_h\big)-(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla E_\delta^{n+1},\nabla\psi_h\big)\\&\quad-(1-\theta)\frac{\chi_1\lambda^*\Delta t}{\theta_f}\big(Kd_t^2\mathrm{div}E_{\boldsymbol\tau}^{n+1},\nabla\psi_h\big)=\big(R_h^{n+\theta},\psi_h\big),\qquad\forall\psi_h\in W_h,\qquad(3.43)\\&E_{\boldsymbol\tau}^0=0,\quad E_\delta^0=0,\quad E_\varpi^{-1}=0.\qquad(3.44)\end{aligned}$$

Using the definitions of the projection operators $\mathcal{Q}_h,\mathcal{S}_h,\mathcal{R}_h$, we have

$$\begin{aligned}&\gamma\big(\varepsilon(Z_{\boldsymbol\tau}^{n+1}),\varepsilon(\mathbf{v}_h)\big)-\big(G_\delta^{n+1},\mathrm{div}\mathbf{v}_h\big)=\big(F_\delta^{n+1},\mathrm{div}\mathbf{v}_h\big),\qquad\forall\mathbf{v}_h\in\mathbf{V}_h,\qquad(3.45)\\&\chi_3\big(G_\delta^{n+1},\varphi_h\big)+\big(\mathrm{div}Z_{\boldsymbol\tau}^{n+1},\varphi_h\big)+\lambda^*\chi_3\big(d_t\mathrm{div}Z_{\boldsymbol\tau}^{n+1},\mathrm{div}\mathbf{v}_h\big)\\&=\chi_1\big(G_\varpi^{n+\theta},\varphi_h\big)-\big(\mathrm{div}Y_{\boldsymbol\tau}^{n+1},\varphi_h\big)-\lambda^*\chi_3\big(d_t\mathrm{div}Y_{\boldsymbol\tau}^{n+1},\mathrm{div}\mathbf{v}_h\big)\\&\quad+\lambda^*\chi_3\big(\mathbf{R}_h^{n+1},\mathrm{div}\mathbf{v}_h\big)+\chi_1(1-\theta)\Delta t\big(d_t\varpi(t_{n+1}),\varphi_h\big),\qquad\forall\varphi_h\in M_h,\qquad(3.46)\\&\big(d_tG_\varpi^{n+\theta},\psi_h\big)+\frac{1}{\theta_f}\big(K\nabla\hat{Z}_p^{n+1},\nabla\psi_h\big)-(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla Z_\delta^{n+1},\nabla\psi_h\big)\\&\quad-(1-\theta)\frac{\chi_1\lambda^*\Delta t}{\theta_f}\big(Kd_t^2\nabla\mathrm{div}Z_{\boldsymbol\tau}^{n+1},\nabla\psi_h\big)=\big(R_h^{n+\theta},\psi_h\big),\qquad\forall\psi_h\in W_h.\qquad(3.47)\end{aligned}$$

Taking $\mathbf{v}_h=d_tZ_{\boldsymbol\tau}^{n+1}$ in (3.45), $\varphi_h=G_\delta^{n+1}$ after applying the difference operator d_t to (3.46) and

$$\psi_h=\hat{Z}_p^{n+1}=F_p^{n+1}-Y_p^{n+1}+\chi_1G_\delta^{n+1}+\chi_2G_\varpi^{n+\theta}+\lambda^*\chi_1d_t\mathrm{div}Z_{\boldsymbol\tau}^{n+1}$$

in (3.47), we have

$$\begin{aligned}&\gamma\big(\varepsilon(Z_{\boldsymbol\tau}^{n+1}),d_t\varepsilon(Z_{\boldsymbol\tau}^{n+1})\big)-\big(G_\delta^{n+1},\mathrm{div}d_tZ_{\boldsymbol\tau}^{n+1}\big)=\big(F_\delta^{n+1},\mathrm{div}d_tZ_{\boldsymbol\tau}^{n+1}\big),\qquad(3.48)\\&\chi_3\big(d_tG_\delta^{n+1},G_\delta^{n+1}\big)+\big(d_t\mathrm{div}Z_{\boldsymbol\tau}^{n+1},G_\delta^{n+1}\big)+\lambda^*\chi_3\big(d_t^2\mathrm{div}Z_{\boldsymbol\tau}^{n+1},G_\delta^{n+1}\big)\\&=\chi_1\big(d_tG_\varpi^{n+\theta},G_\delta^{n+1}\big)-\big(d_t\mathrm{div}Y_{\boldsymbol\tau}^{n+1},G_\delta^{n+1}\big)-\lambda^*\chi_3\big(d_t^2\mathrm{div}Y_{\boldsymbol\tau}^{n+1},G_\delta^{n+1}\big)\\&\quad+\lambda^*\chi_3\big(\mathbf{R}_h^{n+1},G_\delta^{n+1}\big)+\chi_1(1-\theta)\Delta t\big(d_t\varpi(t_{n+1}),G_\delta^{n+1}\big),\qquad(3.49)\\&\big(d_tG_\varpi^{n+\theta},\hat{Z}_p^{n+1}\big)-(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla Z_\delta^{n+1},\nabla\hat{Z}_p^{n+1}\big)+\frac{1}{\theta_f}\big(K\nabla\hat{Z}_p^{n+1},\nabla\hat{Z}_p^{n+1}\big)\\&\quad-(1-\theta)\frac{\chi_1\lambda^*\Delta t}{\theta_f}\big(Kd_t^2\nabla\mathrm{div}Z_{\boldsymbol\tau}^{n+1},\nabla\hat{Z}_p^{n+1}\big)=\big(R_h^{n+\theta},\hat{Z}_p^{n+1}\big).\qquad(3.50)\end{aligned}$$

Using (3.48)-(3.50) and (3.26), we obtain

$$\begin{aligned}&\frac{\gamma\Delta t}{2}\big\|d_t\varepsilon(Z_{\boldsymbol\tau}^{n+1})\big\|_{L^2(\Omega)}^2+\frac{\gamma}{2}d_t\big\|\varepsilon(Z_{\boldsymbol\tau}^{n+1})\big\|_{L^2(\Omega)}^2+\frac{\chi_3\Delta t}{2}\big\|d_tG_\delta^{n+1}\big\|_{L^2(\Omega)}^2\\&\quad+\frac{\chi_3}{2}d_t\big\|G_\delta^{n+1}\big\|_{L^2(\Omega)}^2+\lambda^*\chi_3\big(d_t^2\mathrm{div}Z_{\boldsymbol\tau}^{n+1},G_\delta^{n+1}\big)+\frac{\chi_2\Delta t}{2}\big\|d_tG_\varpi^{n+1}\big\|_{L^2(\Omega)}^2\\&\quad+\frac{\chi_2}{2}d_t\big\|G_\varpi^{n+\theta}\big\|_{L^2(\Omega)}^2+\big(d_tG_\varpi^{n+\theta},\lambda^*\chi_1d_t\mathrm{div}Z_{\boldsymbol\tau}^{n+1}\big)+\frac{1}{\theta_f}\big(K\nabla\hat{Z}_p^{n+1},\nabla\hat{Z}_p^{n+1}\big)\end{aligned}$$

$$
\begin{aligned}
&= \left(F_\delta^{n+1}, \mathrm{div}\, d_t Z_{\boldsymbol{\tau}}^{n+1}\right) - \left(d_t \mathrm{div}\, Y_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) - \lambda^* \chi_3 \left(d_t^2 \mathrm{div}\, Y_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) \\
&\quad + \chi_1 (1-\theta) \Delta t \left(d_t \varpi(t_{n+1}), G_\delta^{n+1}\right) + (1-\theta) \frac{\chi_1 \Delta t}{\theta_f} \left(K d_t \nabla Z_\delta^{n+1}, \nabla \hat{Z}_p^{n+1}\right) \\
&\quad + (1-\theta) \frac{\chi_1 \lambda^* \Delta t}{\theta_f} \left(K d_t^2 \nabla \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, \nabla \hat{Z}_p^{n+1}\right) + \left(R_h^{n+\theta}, \hat{Z}_p^{n+1}\right) \\
&\quad + \left(d_t G_\varpi^{n+\theta}, Y_p^{n+1} - F_p^{n+1}\right) + \lambda^* \chi_3 \left(\mathbf{R}_h^{n+1}, G_\delta^{n+1}\right).
\end{aligned} \tag{3.51}
$$

Using the equality

$$
d_t G_q^n = \chi_1 d_t G_\varpi^n - \chi_3 d_t G_\delta^n - \lambda^* \chi_3 d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^n
$$

and (3.51), we have

$$
\begin{aligned}
&\lambda^* \chi_3 \left(d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) + \left(d_t G_\varpi^n, \lambda^* \chi_1 d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}\right) \\
&= \lambda^* \chi_3 d_t \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) - \lambda^* \chi_3 \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^n, d_t G_\delta^{n+1}\right) + \left(d_t G_\varpi^n, \lambda^* \chi_1 d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}\right) \\
&= \lambda^* \chi_3 d_t \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) - \lambda^* \chi_3 \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^n, d_t G_\delta^{n+1}\right) + \lambda^* \chi_3 \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t G_\delta^{n+1}\right) \\
&\quad - \lambda^* \chi_3 \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t G_\delta^{n+1}\right) + \left(d_t G_\varpi^n, \lambda^* \chi_1 d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}\right) \\
&= \lambda^* \chi_3 d_t \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) + \lambda^* \chi_3 \Delta t \left(d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t G_\delta^{n+1}\right) \\
&\quad + \left(\lambda^* d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, \chi_1 d_t G_\varpi^n - \chi_3 d_t G_\delta^{n+1}\right) \\
&= \lambda^* \chi_3 d_t \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) + \lambda^* \chi_3 \Delta t \left(d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t G_\delta^{n+1}\right) \\
&\quad + \left(\lambda^* d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1} + \lambda^* \chi_3 d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}\right) \\
&= \lambda^* \chi_3 d_t \left(d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) + \lambda^* \chi_3 \Delta t \left(d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t G_\delta^{n+1}\right) \\
&\quad + \left(\lambda^* d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}\right) + \left(\lambda^* d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}, \lambda^* \chi_3 d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1}\right).
\end{aligned} \tag{3.52}
$$

Taking $\mathbf{v}_h = d_t Z_{\boldsymbol{\tau}}^{n+1}$ in (3.45), we obtain

$$
\gamma \left(\varepsilon\left(d_t Z_{\boldsymbol{\tau}}^{n+1}\right), d_t^2 \varepsilon\left(Z_{\boldsymbol{\tau}}^{n+1}\right)\right) - \left(d_t G_\delta^{n+1}, \mathrm{div}\, d_t^2 Z_{\boldsymbol{\tau}}^{n+1}\right) = \left(d_t F_\delta^{n+1}, \mathrm{div}\, d_t^2 Z_{\boldsymbol{\tau}}^{n+1}\right). \tag{3.53}
$$

Using (3.52), (3.53) and (3.51), we get

$$
\begin{aligned}
&\frac{\gamma \Delta t}{2} \left\| d_t \varepsilon\left(Z_{\boldsymbol{\tau}}^{n+1}\right) \right\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} d_t \left\| \varepsilon\left(Z_{\boldsymbol{\tau}}^{n+1}\right) \right\|_{L^2(\Omega)}^2 + \frac{\chi_3 \Delta t}{2} \left\| d_t G_\delta^{n+1} \right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\chi_3}{2} d_t \left\| \lambda^* d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1} + G_\delta^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\chi_3 \Delta t}{2} \left\| \lambda^* d_t^2 \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1} \right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\gamma \chi_3 \lambda^* \Delta t}{2} d_t \left\| d_t \varepsilon\left(Z_{\boldsymbol{\tau}}^{n+1}\right) \right\|_{L^2(\Omega)}^2 + \frac{\gamma \chi_3 \lambda^* (\Delta t)^2}{2} \left\| d_t^2 \varepsilon\left(Z_{\boldsymbol{\tau}}^{n+1}\right) \right\|_{L^2(\Omega)}^2 \\
&\quad + \lambda^* \left\| d_t \mathrm{div}\, Z_{\boldsymbol{\tau}}^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\chi_2 \Delta t}{2} \left\| d_t G_\varpi^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\chi_2}{2} d_t \left\| G_\varpi^{n+\theta} \right\|_{L^2(\Omega)}^2 \\
&\quad - \chi_3 \lambda^* \Delta t \left(d_t F_\delta^{n+1}, \mathrm{div}\, d_t^2 Z_{\boldsymbol{\tau}}^{n+1}\right) + \frac{1}{\theta_f} \left(K \nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_p^{n+1}\right) \\
&= \left(F_\delta^{n+1}, \mathrm{div}\, d_t Z_{\boldsymbol{\tau}}^{n+1}\right) - \left(d_t \mathrm{div}\, Y_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right) - \lambda^* \chi_3 \left(d_t^2 \mathrm{div}\, Y_{\boldsymbol{\tau}}^{n+1}, G_\delta^{n+1}\right)
\end{aligned}
$$

$$+\chi_1(1-\theta)\Delta t\big(d_t\varpi(t_{n+1}),G_\delta^{n+1}\big)+(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla Z_\delta^{n+1},\nabla\hat{Z}_p^{n+1}\big)$$
$$+(1-\theta)\frac{\chi_1\lambda^*\Delta t}{\theta_f}\big(Kd_t^2\nabla\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1},\nabla\hat{Z}_p^{n+1}\big)+\big(R_h^{n+\theta},\hat{Z}_p^{n+1}\big)$$
$$+\big(d_tG_\varpi^{n+\theta},Y_p^{n+1}-F_p^{n+1}\big)+\lambda^*\chi_3\big(\mathbf{R}_h^{n+1},G_\delta^{n+1}\big). \tag{3.54}$$

Applying the summation operator $\Delta t\sum_{n=0}^{l}$ to both sides of (3.54), the (3.40) holds. The proof is complete. □

Theorem 3.1. *Let $\{(\boldsymbol{\tau}_h^n,\delta_h^n,\varpi_h^n)\}_{n\geq 0}$ be defined by the (MFEA), then there holds*

$$\max_{0\leq n\leq l}\Big[\sqrt{\gamma}\|\varepsilon(Z_{\boldsymbol{\tau}}^{n+1})\|_{L^2(\Omega)}+\sqrt{\chi_2}\|G_\varpi^{n+\theta}\|_{L^2(\Omega)}+\sqrt{\chi_3}\|\lambda^*d_t\mathrm{div}Z_{\boldsymbol{\tau}}^{l+1}+G_\delta^{l+1}\|_{L^2(\Omega)}\Big]$$
$$+\left[\Delta t\sum_{n=1}^{l}\frac{K}{\theta_f}\|\nabla\hat{Z}_p^{n+1}\|^2_{L^2(\Omega)}\right]^{\frac{1}{2}}\leq\hat{C}_1(T)\Delta t+\hat{C}_2(T)h^2, \tag{3.55}$$

provided that $\Delta t=\mathcal{O}(h^2)$ when $\theta=0$ and $\Delta t>0$ when $\theta=1$. Here

$$\hat{C}_1(T)=\hat{C}\|\varpi_t\|_{L^2((0,T);L^2(\Omega))}+\hat{C}\|\varpi_{tt}\|_{L^2((0,T);H^1(\Omega)')}$$
$$+\hat{C}\|\mathrm{div}\boldsymbol{\tau}_{tt}\|_{L^2((0,T);H^1(\Omega)')}, \tag{3.56}$$
$$\hat{C}_2(T)=\hat{C}\|\delta_t\|_{L^2((0,T);H^2(\Omega))}+\hat{C}\|\delta\|_{L^\infty((0,T);H^2(\Omega))}$$
$$+\hat{C}\|\boldsymbol{\tau}\|_{L^2((0,T);H^3(\Omega))}+\hat{C}\|\nabla\cdot\boldsymbol{\tau}_t\|_{L^2((0,T);H^2(\Omega))}. \tag{3.57}$$

Proof. Using (3.40) and the fact of $Z_{\boldsymbol{u}}^0=\mathbf{0}$, $Z_\delta^0=0$ and $Z_\varpi^{-1}=0$, we have

$$\mathcal{E}_h^l+\Delta t\sum_{n=0}^{l}\bigg[\frac{\Delta t}{2}\Big(\chi_3\big\|\lambda^*d_t^2\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1}\big\|^2_{L^2(\Omega)}+\gamma\big\|d_t\varepsilon\big(Z_{\boldsymbol{\tau}}^{n+1}\big)\big\|^2_{L^2(\Omega)}+\chi_3\big\|d_tG_\delta^{n+1}\big\|^2_{L^2(\Omega)}$$
$$+\chi_2\big\|d_tG_\varpi^{n+\theta}\big\|^2_{L^2(\Omega)}+\gamma\chi_3\lambda^*\Delta t\big\|d_t^2\varepsilon\big(Z_{\boldsymbol{\tau}}^{n+1}\big)\big\|^2_{L^2(\Omega)}\Big)$$
$$+\lambda^*\big\|d_t\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1}\big\|^2_{L^2(\Omega)}+\frac{1}{\theta_f}\big(K\nabla\hat{Z}_p^{n+1},\nabla\hat{Z}_p^{n+1}\big)\bigg]$$
$$=\Phi_1+\Phi_2+\Phi_3+\Phi_4+\Phi_5+\Phi_6+\Phi_7+\Phi_8+\Phi_9, \tag{3.58}$$

where

$$\Phi_1=\chi_3\lambda^*(\Delta t)^2\sum_{n=0}^{l}\big(d_tF_\delta^{n+1},\mathrm{div}d_t^2Z_{\boldsymbol{\tau}}^{n+1}\big),$$
$$\Phi_2=\Delta t\sum_{n=0}^{l}\big[\big(F_\delta^{n+1},\mathrm{div}d_tZ_{\boldsymbol{\tau}}^{n+1}\big)-\big(\mathrm{div}d_tY_{\boldsymbol{\tau}}^{n+1},G_\delta^{n+1}\big)\big],$$
$$\Phi_3=\Delta t\sum_{n=0}^{l}\chi_1(1-\theta)\Delta t\big(d_t^2\varpi(t_{n+1}),G_\delta^{n+1}\big),$$

$$\Phi_4=\Delta t\sum_{n=0}^{l}(1-\theta)\frac{\chi_1\Delta t}{\theta_f}\big(Kd_t\nabla Z_\delta^{n+1},\nabla\hat{Z}_p^{n+1}\big),$$

$$\Phi_5=\Delta t\sum_{n=0}^{l}\big(d_tG_\varpi^{n+\theta},Y_p^{n+1}-F_p^{n+1}\big),$$

$$\Phi_6=\Delta t\sum_{n=0}^{l}(1-\theta)\frac{\chi_1\lambda^*\Delta t}{\theta_f}\big(Kd_t^2\nabla\mathrm{div}\,Z_{\boldsymbol{\tau}}^{n+1},\nabla\hat{Z}_p^{n+1}\big),$$

$$\Phi_7=\Delta t\sum_{n=0}^{l}-\lambda^*\chi_3\big(d_t^2\mathrm{div}\,Y_{\boldsymbol{\tau}}^{n+1},G_\delta^{n+1}\big),$$

$$\Phi_8=\Delta t\sum_{n=0}^{l}\big(R_h^{n+\theta},\hat{Z}_p^{n+1}\big),$$

$$\Phi_9=\Delta t\sum_{n=0}^{l}\lambda^*\chi_3\big(\mathbf{R}_h^{n+1},G_\delta^{n+1}\big).$$

Next, we estimate each term of (3.58). As for the boundness of Φ_2,Φ_3,Φ_4 and Φ_8, one can refer to [11], here we omit the details. For Φ_1 and Φ_7, using Cauchy-Schwarz inequality and Young inequality, we have

$$\begin{aligned}
\Phi_1&=\chi_3\lambda^*(\Delta t)^2\sum_{n=0}^{l}\big(d_tF_\delta^{n+1},\mathrm{div}\,d_t^2Z_{\boldsymbol{\tau}}^{n+1}\big)\\
&\le\chi_3\lambda^*(\Delta t)^2\sum_{n=0}^{l}\big\|d_tF_\delta^{n+1}\big\|_{L^2(\Omega)}\big\|\mathrm{div}\,d_t^2Z_{\boldsymbol{\tau}}^{n+1}\big\|_{L^2(\Omega)}\\
&\le(\Delta t)^2\sum_{n=0}^{l}\left[\chi_3\big\|d_tF_\delta^{n+1}\big\|_{L^2(\Omega)}^2+\frac{\chi_3(\lambda^*)^2}{4}\big\|\mathrm{div}\,d_t^2Z_{\boldsymbol{\tau}}^{n+1}\big\|_{L^2(\Omega)}^2\right],
\end{aligned}\tag{3.59}$$

$$\begin{aligned}
\Phi_7&=\Delta t\sum_{n=0}^{l}-\lambda^*\chi_3\big(d_t^2\mathrm{div}\,Y_{\boldsymbol{\tau}}^{n+1},G_\delta^{n+1}\big)\\
&\le\lambda^*\chi_3\Delta t\sum_{n=0}^{l}\big\|d_t^2\mathrm{div}\,Y_{\boldsymbol{\tau}}^{n+1}\big\|_{L^2(\Omega)}\big\|G_\delta^{n+1}\big\|_{L^2(\Omega)}\\
&\le\lambda^*\chi_3\Delta t\sum_{n=0}^{l}\left[\frac{1}{2}\big\|d_t^2\mathrm{div}\,Y_{\boldsymbol{\tau}}^{n+1}\big\|_{L^2(\Omega)}^2+\frac{1}{2}\big\|G_\delta^{n+1}\big\|_{L^2(\Omega)}^2\right].
\end{aligned}\tag{3.60}$$

As for the boundness of Φ_5, using the Cauchy-Schwarz inequality and Young inequality, we get

$$\begin{aligned}
\Phi_5&=\Delta t\sum_{n=0}^{l}\big(d_tG_\varpi^{n+\theta},Y_p^{n+1}-F_p^{n+1}\big)\\
&=\Delta t\left[\frac{1}{\Delta t}\big(G_\varpi^{l+\theta},Y_p^{l+1}-F_p^{l+1}\big)-\sum_{n=1}^{l}\big(G_\varpi^{n+\theta},d_tY_p^{n+1}-d_tF_p^{n+1}\big)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \Delta t\left[\frac{1}{\Delta t}\|G_{\varpi}^{l+\theta}\|_{L^2(\Omega)}\|Y_p^{l+1}-F_p^{l+1}\|_{L^2(\Omega)}+\sum_{n=0}^{l}\|G_{\varpi}^{n+\theta}\|_{L^2(\Omega)}\|d_tY_p^{n+1}-d_tF_p^{n+1}\|_{L^2(\Omega)}\right]\\
&\leq \Delta t\left[\frac{1}{\Delta t}\left(\frac{\chi_2}{4}\|G_{\varpi}^{l+\theta}\|^2_{L^2(\Omega)}+\frac{1}{\chi_2}\|Y_p^{l+1}-F_p^{l+1}\|^2_{L^2(\Omega)}\right)+\chi_2\sum_{n=1}^{l}\|G_{\varpi}^{n+\theta}\|_{L^2(\Omega)}\right.\\
&\qquad\left.+\frac{1}{2\chi_2}\sum_{n=1}^{l}\left(\|d_tF_p^{n+1}\|^2_{L^2(\Omega)}+\|d_tY_p^{n+1}\|^2_{L^2(\Omega)}\right)\right]. \qquad (3.61)
\end{aligned}$$

When $\theta=0$, using Cauchy-Schwarz inequality and Young inequality for Φ_6, we get

$$\begin{aligned}
\Phi_6&=\Delta t\sum_{n=0}^{l}\frac{\chi_1\lambda^*\Delta t}{\theta_f}\left(Kd_t^2\nabla\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1},\nabla\hat{Z}_p^{n+1}\right)\\
&\leq(\Delta t)^2\sum_{n=0}^{l}\frac{\chi_1\lambda^*K_2}{h\theta_f}\|d_t^2\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1}\|_{L^2(\Omega)}\|\nabla\hat{Z}_p^{n+1}\|_{L^2(\Omega)}\\
&\leq(\Delta t)^2\sum_{n=0}^{l}\left[\frac{\chi_1^2(\lambda^*)^2K_2^2\Delta t}{h^2\theta_fK_1}\|d_t^2\mathrm{div}Z_{\boldsymbol{\tau}}^{n+1}\|^2_{L^2(\Omega)}+\frac{K_1}{4\theta_f\Delta t}\|\nabla\hat{Z}_p^{n+1}\|_{L^2(\Omega)}\right]. \qquad (3.62)
\end{aligned}$$

As for Φ_9, using the fact of

$$\|\mathbf{R}_h^{n+1}\|^2_{H^1(\Omega)'}\leq\frac{\Delta t}{3}\int_{t_n}^{t_{n+1}}\|\mathrm{div}\boldsymbol{\tau}_{tt}\|^2_{H^1(\Omega)'}dt,$$

the Cauchy-Schwarz inequality and Young inequality, we get

$$\begin{aligned}
&\Delta t\sum_{n=0}^{l}\lambda^*\chi_3\left(\mathbf{R}_h^{n+1},G_{\delta}^{n+1}\right)\\
&\leq\lambda^*\chi_3\Delta t\sum_{n=0}^{l}\left[\|\mathbf{R}_h^{n+1}\|^2_{H^1(\Omega)'}+\frac{1}{4}\|G_{\delta}^{n+1}\|^2_{L^2(\Omega)}\right]\\
&\leq\lambda^*\chi_3\sum_{n=0}^{l}\Delta t\left[\frac{\Delta t}{3}\|\mathrm{div}\boldsymbol{\tau}_{tt}\|^2_{L^2(t_n,t_{n+1};H^1(\Omega)')}+\frac{1}{4}\|G_{\delta}^{n+1}\|^2_{L^2(\Omega)}\right]. \qquad (3.63)
\end{aligned}$$

Adding (3.59)-(3.63) and applying the discrete Gronwall inequality (cf. [27]), we have

$$\begin{aligned}
&\gamma\|\varepsilon(Z_{\boldsymbol{\tau}}^{l+1})\|^2_{L^2(\Omega)}+\chi_2\|G_{\varpi}^{l+\theta}\|^2_{L^2(\Omega)}+\chi_3\|\lambda^*d_t\mathrm{div}Z_{\boldsymbol{\tau}}^{l+1}+G_{\delta}^{l+1}\|^2_{L^2(\Omega)}\\
&\quad+\Delta t\sum_{n=0}^{l}\frac{K_1}{\theta_f}\|\nabla\hat{Z}_p^{n+1}\|^2_{L^2(\Omega)}\\
&\leq\hat{C}\left[\frac{4\gamma\chi_1^2}{\beta_1^2}\|\varpi_t\|^2_{L^2((0,T);L^2(\Omega))}+\frac{\theta_f(\Delta t)^2}{3K_1}\|\varpi_{tt}\|^2_{L^2((0,T);H^1(\Omega)')}\right.\\
&\qquad+\frac{\lambda^*\chi_3(\Delta t)^2}{3}\|\mathrm{div}\boldsymbol{\tau}_{tt}\|^2_{L^2((0,T);H^1(\Omega)')}+\|F_{\delta}^{l+1}\|^2_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&+\Delta t\sum_{n=0}^{l}\|d_tF_\delta^{l+1}\|_{L^2(\Omega)}^2+\Delta t\sum_{n=0}^{l}\|d_t\mathrm{div}Y_{\boldsymbol{\tau}}^{l+1}\|_{L^2(\Omega)}^2\Big]\\
\leq&\hat{C}(\Delta t)^2\left(\|\varpi_t\|^2_{L^2((0,T);L^2(\Omega))}+\|\varpi_{tt}\|^2_{L^2((0,T);H^1(\Omega)')}+\|\mathrm{div}\boldsymbol{\tau}_{tt}\|^2_{L^2((0,T);H^1(\Omega)')}\right)\\
&+\hat{C}h^4\left(\|\delta_t\|^2_{L^2((0,T);H^2(\Omega))}+\|\delta\|^2_{L^\infty((0,T);H^2(\Omega))}+\|\mathrm{div}\boldsymbol{\tau}_t\|^2_{L^2((0,T);H^2(\Omega))}\right) \qquad (3.64)
\end{aligned}
$$

provided that

$$
\Delta t\leq\min\left\{\frac{\theta_f K_1\beta_1^2h^2}{4\gamma\chi_1^2K_2^2},\frac{\theta_f\chi_3h^2K_1}{4\chi_1^2K_2^2}\right\}
$$

when $\theta=0$, and it holds for all $\Delta t>0$ when $\theta=1$. From (3.64), we deduce that (3.55) holds. The proof is complete. □

Theorem 3.2. *The solution of the (MFEA) satisfies the following error estimates:*

$$
\begin{aligned}
&\max_{0\leq n\leq N}\Big[\sqrt{\gamma}\|\nabla\left(\boldsymbol{\tau}(t_{n+1})-\boldsymbol{\tau}_h^{n+1}\right)\|_{L^2(\Omega)}\\
&\qquad+\sqrt{\chi_2}\|\varpi(t_{n+1})-\varpi_h^{n+1}\|_{L^2(\Omega)}+\sqrt{\chi_3}\|\delta(t_{n+1})-\delta_h^{n+1}\|_{L^2(\Omega)}\Big]\\
\leq&\check{C}_1(T)\Delta t+\check{C}_2(T)h^2, \qquad (3.65)
\end{aligned}
$$

$$
\left(\Delta t\sum_{n=0}^{N}\frac{K}{\theta_f}\|\nabla p(t_{n+1})-\nabla p_h^{n+1}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\leq\check{C}_1(T)\Delta t+\check{C}_2(T)h, \qquad (3.66)
$$

provided that $\Delta t=\mathcal{O}(h^2)$ when $\theta=0$ and $\Delta t\geq 0$ when $\theta=1$. Here

$$
\begin{aligned}
&\check{C}_1(T)=\hat{C}_1(T),\\
&\check{C}_2(T)=\hat{C}_2(T)+\|\delta\|_{L^\infty((0,T);H^2(\Omega))}+\|\varpi\|_{L^\infty((0,T);H^2(\Omega))}+\|\nabla\boldsymbol{u}\|_{L^\infty((0,T);H^2(\Omega))}.
\end{aligned}
$$

Proof. The above estimates follow immediately from an application of the triangle inequality on

$$
\begin{aligned}
&\boldsymbol{\tau}(t_n)-\boldsymbol{\tau}_h^n=Y_{\boldsymbol{\tau}}^n+Z_{\boldsymbol{\tau}}^n, && \delta(t_n)-\delta_h^n=Y_\delta^n+Z_\delta^n=F_\delta^n+G_\delta^n,\\
&\varpi(t_n)-\varpi_h^n=Y_\varpi^n+Z_\varpi^n=F_\varpi^n+G_\varpi^n, && p(t_n)-p_h^n=Y_p^n+Z_p^n=F_p^n+G_p^n.
\end{aligned}
$$

and appealing to (3.36)-(3.38) and Theorem 3.1. The proof is complete. □

4 Numerical tests

Test 1. Let $\Omega=[0,1]\times[0,1]$,

$$
\begin{aligned}
&\Gamma_1=\{(1,x_2):0\leq x_2\leq 1\}, \quad \Gamma_2=\{(x_1,0):0\leq x_1\leq 1\},\\
&\Gamma_3=\{(0,x_2):0\leq x_2\leq 1\}, \quad \Gamma_4=\{(x_1,1):0\leq x_1\leq 1\},
\end{aligned}
$$

and $T=1$. The source functions are as follows:

$$\mathbf{F}=\lambda^*\pi^2\big(\sin(\pi x_1),\sin(\pi x_2)\big)'+(\beta+\gamma)\pi^2 t\big(\sin(\pi x_1),\sin(\pi x_2)\big)'$$
$$+b_0 t\pi\cos(\pi x_1+\pi x_2)(1,1)',$$
$$\phi=a_0\sin(\pi x_1+\pi x_2)+\frac{2K}{\theta_f}t\pi^2\sin(\pi x_1+\pi x_2)$$
$$+b_0\pi\big(\cos(\pi x_1)+\cos(\pi x_2)\big).$$

The boundary and initial conditions are

$$\begin{aligned}
&p=t\sin(\pi x_1+\pi x_2) && \text{on } \partial\Omega_T,\\
&\tau_1=t\sin(\pi x_1) && \text{on } \Gamma_j\times(0,T),\quad j=1,3,\\
&\tau_2=t\sin(\pi x_2) && \text{on } \Gamma_j\times(0,T),\quad j=2,4,\\
&\lambda^*\mathrm{div}\boldsymbol{\tau}_t\mathbf{n}+\sigma\mathbf{n}-b_0 p\mathbf{n}=\mathbf{F_1} && \text{on } \partial\Omega_T,\\
&\boldsymbol{\tau}(x,0)=\mathbf{0},\quad p(x,0)=0 && \text{in } \Omega,
\end{aligned}$$

where

$$\mathbf{F}_1(x,t)=\lambda^*\pi\big(\cos(\pi x_1)+\cos(\pi x_2)\big)(1,1)'+\gamma\pi t\big(\cos(\pi x_1),\cos(\pi x_2)\big)'$$
$$+\beta\pi t\big(\cos(\pi x_1)+\cos(\pi x_2)\big)(1,1)'-b_0 t\sin(\pi x_1+\pi x_2)(1,1)'.$$

The exact solution of this problem is

$$\boldsymbol{\tau}(x,t)=t\big(\sin(\pi x_1),\sin(\pi x_2)\big)',\quad p(x,t)=t\sin(\pi x_1+\pi x_2).$$

As for the convergence order of time discretization, we define

$$\rho_{h,\Delta t}=\frac{\big\|v^{h,\Delta t}-v^{h,\Delta t/2}\big\|_{L^2}}{\big\|v^{h,\Delta t/2}-v^{h,\Delta t/4}\big\|_{L^2}},$$

where $v=\boldsymbol{\tau},p$. In particular, $\rho_{h,\Delta t}\approx 2$ when the corresponding order of convergence in time is of $\mathcal{O}(\Delta t)$, one can refer to [20].

Test 2. The Ω and T are the same as ones of Test 1. And the source functions are

$$\mathbf{F}=\lambda^* e^t(\sin x_1,\sin x_2)'+(\beta+\gamma)e^t(\sin x_1,\sin x_2)'$$
$$+b_0 t\pi\big(\cos(\pi x_1)\sin(\pi x_2),\sin(\pi x_1)\cos(\pi x_2)\big)',$$
$$\phi=a_0\sin(\pi x_1)\sin(\pi x_2)+\frac{2K}{\theta_f}\pi^2 t\sin(\pi x_1)\sin(\pi x_2)$$
$$+b_0 e^t(\cos x_1+\cos x_2).$$

The boundary and initial conditions are

$$p=t\sin(\pi x_1)\sin(\pi x_2)\quad \text{on } \partial\Omega_T,$$

$$\begin{aligned}
&\tau_1 = e^t \sin x_1 && \text{on } \Gamma_j \times (0,T), \quad j=1,3,\\
&\tau_2 = e^t \sin x_2 && \text{on } \Gamma_j \times (0,T), \quad j=2,4,\\
&\lambda^* \mathrm{div} \boldsymbol{\tau}_t \mathbf{n} + \sigma \mathbf{n} - b_0 p \mathbf{n} = \mathbf{F_1} && \text{on } \partial\Omega_T,\\
&\boldsymbol{\tau}(x,0) = \mathbf{0}, \quad p(x,0) = 0 && \text{in } \Omega,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{F}_1(x,t) = &\lambda^*(\cos x_1 + \cos x_2)(n_1,n_2)' e^t + \gamma e^t (\cos x_1, \cos x_2)' \\
&+ \beta e^t (\cos x_1 + \cos x_2)(n_1,n_2)' - b_0 t \sin(\pi x_1)\sin(\pi x_2)(n_1,n_2)'.
\end{aligned}$$

The exact solution of this problem is

$$\boldsymbol{\tau}(x,t) = e^t (\sin x_1, \sin x_2)', \quad p(x,t) = t\sin(\pi x_1)\sin(\pi x_2).$$

Figs. 1 and 2 show the numerical solution of displacement $(\tau_1)_h^{n+1}$ and $(\tau_2)_h^{n+1}$ at the terminal time T with the parameters of Table 1 of Test 1. Figs. 3 and 4 show the numerical solution of pressure p_h^{n+1} and the arrow plot of the computed displacement $\boldsymbol{\tau}_h^{n+1}$ with the parameters of Table 1 of Test 1, respectively. Tables 5 and 6, and Figs. 5-8 of Test 2 have similar results as ones of Test 1 for the case of $\nu = 0.4$, which are consistent with the theoretical results.

Table 2 displays the spatial errors and convergence rates of displacement $\boldsymbol{\tau}_h$ and the pressure p_h in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm with the parameters of Table 1 of Test 1. Table 3 shows the convergence rate of time discretization is 1 when $\rho_{h,\Delta t} \approx 2$. The results are consistent with the theoretical results.

Table 1: Values of parameters.

Parameters	Description	Values
λ^*	Coefficient of secondary consolidation	1e-5
ν	Poisson ratio	0.25
b_0	Biot-Willis constant	1e-5
E	Young's modulus	25
β	Lamé constant	10
K	Permeability tensor	(1e-3)**I**
γ	Lamé constant	10
a_0	Constrained specific storage coefficient	0.2

Table 2: Spatial errors and convergence rates (CR) of $\boldsymbol{\tau}_h$ and p_h.

h	$\Vert\boldsymbol{\tau}-\boldsymbol{\tau}_h\Vert_{L^2}$	CR	$\Vert\boldsymbol{\tau}-\boldsymbol{\tau}_h\Vert_{H^1}$	CR	$\Vert p-p_h\Vert_{L^2}$	CR	$\Vert p-p_h\Vert_{H^1}$	CR
$h=1/4$	2.6318e-3		7.9301e-2		2.6672e-2		7.3216e-1	
$h=1/8$	3.1932e-4	3.043	1.8635e-2	2.0893	5.6605e-3	2.2363	3.5970e-1	1.0254
$h=1/16$	3.9427e-5	3.0177	4.5654e-3	2.0292	1.3277e-3	2.0920	1.7857e-1	1.0103
$h=1/32$	4.9094e-6	3.0056	1.1336e-3	2.0098	3.2584e-4	2.0267	8.9098e-2	1.0030

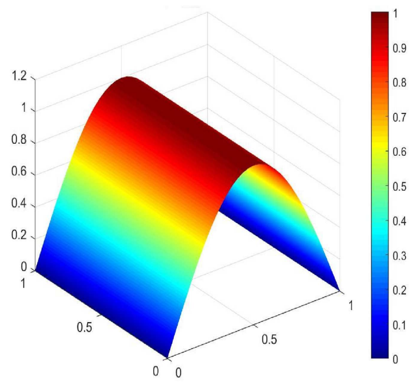

Figure 1: The numerical displacement $(\tau_1)_h^{n+1}$ at the terminal time T with the parameters of Table 1.

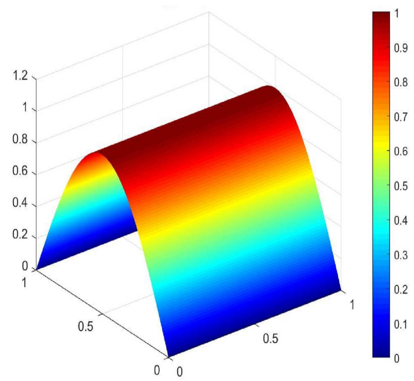

Figure 2: The numerical displacement $(\tau_2)_h^{n+1}$ at the terminal time T with the parameters of Table 1.

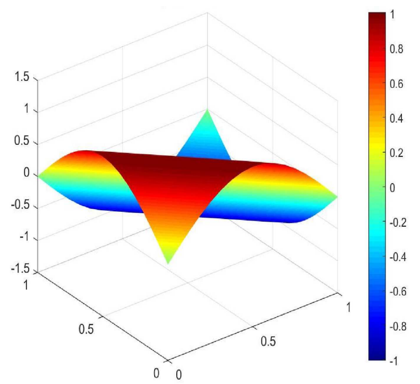

Figure 3: The numerical pressure p_h^{n+1} at the terminal time T with the parameters of Table 1.

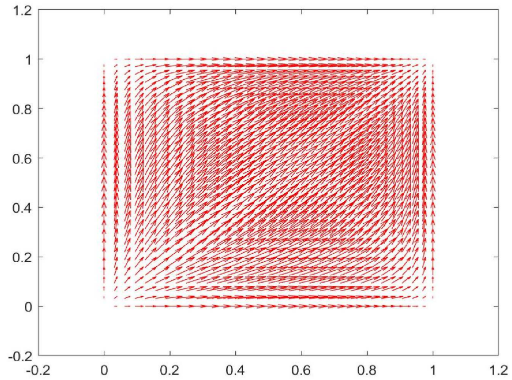

Figure 4: Arrow plot of the computed displacement $\boldsymbol{\tau}_h^{n+1}$ with the parameters of Table 1.

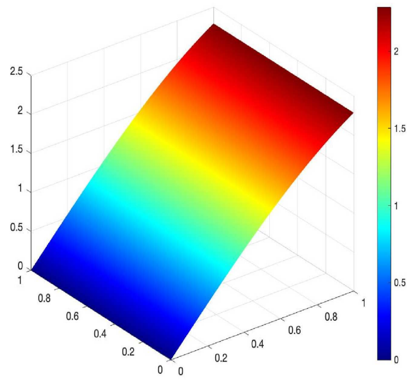

Figure 5: The numerical displacement $(\tau_1)_h^{n+1}$ at the terminal time T with the parameters of Table 4.

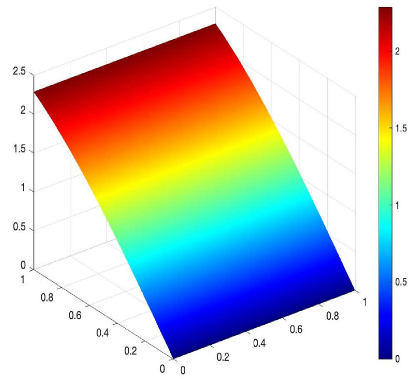

Figure 6: The numerical displacement $(\tau_2)_h^{n+1}$ at the terminal time T with the parameters of Table 4.

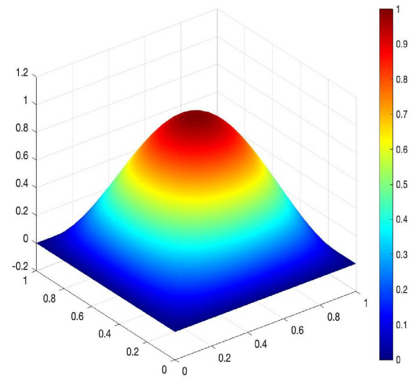

Figure 7: The numerical pressure p_h^{n+1} at the terminal time T with the parameters of Table 4.

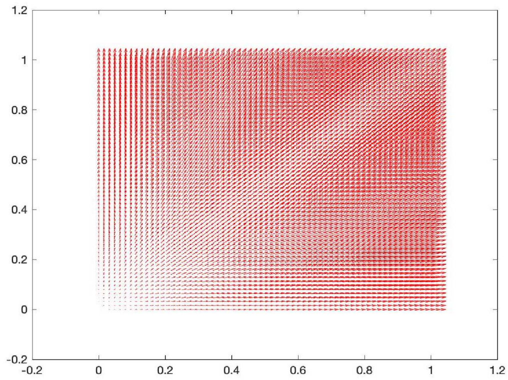

Figure 8: Arrow plot of the computed displacement $\boldsymbol{\tau}_h^{n+1}$ with the parameters of Table 4.

Table 3: Convergence rates of time discretization of Test 1 with $h=1/8$.

Δt	$\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{L^2}$	$\rho_{h,\Delta t}$	$\|p-p_h\|_{L^2}$	$\rho_{h,\Delta t}$
$\Delta t=1/10$	5.1594e-9		4.5958e-5	
$\Delta t=1/20$	2.5796e-9	2.0001	2.3380e-5	1.9657
$\Delta t=1/40$	1.2898e-9	2.0000	1.1793e-5	1.9825
$\Delta t=1/80$	6.4489e-10	2.0000	5.9228e-6	1.9911

Table 4: Values of parameters.

Parameters	Description	Values
λ^*	Coefficient of secondary consolidation	1e-5
ν	Poisson ratio	0.4
b_0	Biot-Willis constant	1e-5
E	Young's modulus	28
β	Lamé constant	40
K	Permeability tensor	(1e-7)**I**
γ	Lamé constant	10
a_0	Constrained specific storage coefficient	0.2

Table 5: Spatial errors and convergence rates of $\boldsymbol{\tau}_h$ and p_h.

h	$\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{L^2}$	CR	$\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{H^1}$	CR	$\|p-p_h\|_{L^2}$	CR	$\|p-p_h\|_{H^1}$	CR
$h=1/4$	3.7346e-4		1.1339e-2		3.5134e-2		9.2770e-1	
$h=1/8$	3.9860e-5	3.2279	2.4370e-3	2.2181	7.4862e-3	2.2306	4.4705e-1	1.0532
$h=1/16$	4.5641e-6	3.1266	5.4843e-4	2.1518	1.7325e-3	2.1114	2.2014e-1	1.0220
$h=1/32$	5.4456e-7	3.0672	1.2859e-4	2.0925	4.2483e-4	2.0279	1.0946e-1	1.0080

Table 6: Convergence rates of time discretization of Test 2 with $h=1/10$.

Δt	$\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{L^2}$	$\rho_{h,\Delta t}$	$\|p-p_h\|_{L^2}$	$\rho_{h,\Delta t}$
$\Delta t=1/10$	8.5908e-10		3.5245e-6	
$\Delta t=1/20$	4.3806e-10	1.9611	1.7407e-6	2.0247
$\Delta t=1/40$	2.2122e-10	1.9802	8.6499e-7	2.0124
$\Delta t=1/80$	1.1116e-10	1.9900	4.3115e-7	2.0062

Test 3. This is a benchmark problem, which occurs "locking phenomenon" (cf. [25]). The domain Ω and time T are the same as ones of Test 1. The source functions are $\mathbf{F}=0$, $\phi=0$, and the boundary and initial conditions are

$$\begin{aligned}
&p=0 && \text{on } \partial\Omega_T,\\
&\tau_1=0 && \text{on } \Gamma_j\times(0,T),\quad j=1,3,\\
&\tau_2=0 && \text{on } \Gamma_j\times(0,T),\quad j=2,4,\\
&\lambda^*\mathrm{div}\boldsymbol{\tau}_t\mathbf{n}+\sigma\mathbf{n}-b_0p\mathbf{I}\mathbf{n}=\mathbf{F}_1:=(\mathbf{0},b_0p)' && \text{on } \partial\Omega_T,\\
&\boldsymbol{\tau}(x,0)=\mathbf{0},\quad p(x,0)=0 && \text{in } \Omega,
\end{aligned}$$

where

$$p=\begin{cases}\sin t, & \text{when}\quad x_1\in[0.2,0.8)\times(0,T),\\ 0, & \text{otherwise.}\end{cases}$$

Figs. 9-11 show the numerical pressure p_h^{n+1} of the original model and the reformulated model and the arrow plot of the computed displacement $\boldsymbol{\tau}_h^{n+1}$ corresponding to the parameters of Table 7 of Test 3, respectively. It is easy to observe that the MFEA has no "locking phenomenon".

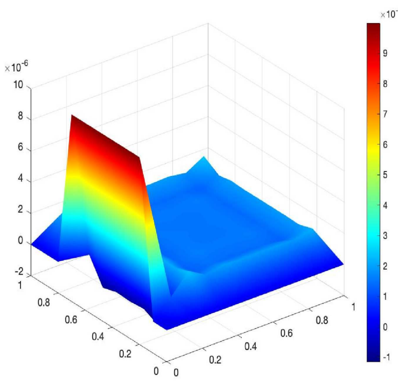

Figure 9: The numerical pressure p_h^{n+1} at $t=0.00001$ with the parameters of Table 7 of the original model.

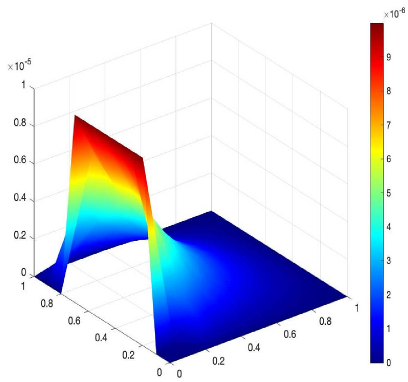

Figure 10: The numerical pressure p_h^{n+1} at $t=0.00001$ with the parameters of Table 7 of the reformulated model.

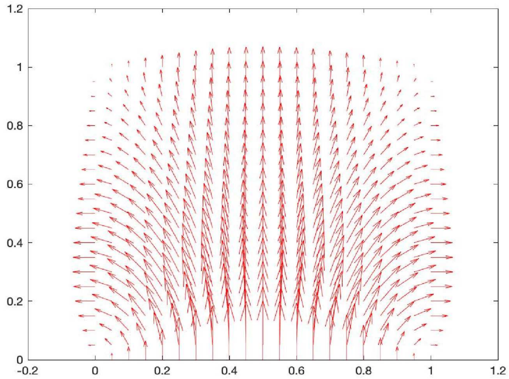

Figure 11: Arrow plot of the computed displacement $\boldsymbol{\tau}_h$ at $t=0.00001$ with the parameters of Table 7 of the reformulated model.

Table 7: Values of parameters.

Parameters	Description	Values
λ^*	Coefficient of secondary consolidation	1e-5
ν	Poisson ratio	0.4
b_0	Biot-Willis constant	1
E	Young's modulus	2.8e3
β	Lamé constant	4e3
K	Permeability tensor	(1e-8)**I**
γ	Lamé constant	1e3
a_0	Constrained specific storage coefficient	0

Test 4. This problem is a real two dimensional footing problem (cf. [14]). The simulation domain is a 100 by 100 meters block of porous soil, and we denote simulation domain by $\Omega=[-50,50]\times[0,100], T=0.01$ s. At the base of this domain the soil is assumed to be fixed while at some centered upper part of the domain a uniform load of intensity $\sigma_0=10^4 N/m^2$ is applied in a strip of length 40m. The whole domain is assumed free to drain. The boundary conditions are given as follows:

$$\begin{aligned}
&p=0 && \text{on } \partial\Omega_T,\\
&\sigma_{xy}=0, \quad \lambda^*(\mathrm{div}\boldsymbol{\tau})_t+\sigma_{yy}=-\sigma_0 && \text{on } \Gamma_1\times(0,T),\\
&\sigma_{xy}=0, \quad \lambda^*(\mathrm{div}\boldsymbol{\tau})_t+\sigma_{yy}=-\sigma_0 && \text{on } \Gamma_2\times(0,T),\\
&\boldsymbol{\tau}=\mathbf{0} && \text{on } \partial\Omega\backslash(\Gamma_1\cup\Gamma_1),
\end{aligned}$$

where

$$\sigma_{xy}=\frac{\gamma}{2}\left(\frac{\partial\tau_1}{\partial x_2}+\frac{\partial\tau_2}{\partial x_1}\right), \quad \sigma_{yy}=\gamma\frac{\partial\tau_2}{\partial x_2}+\beta\left(\frac{\partial\tau_1}{\partial x_1}+\frac{\partial\tau_2}{\partial x_2}\right),$$

and

$$\Gamma_1 = \{(x_1,x_2) \in \partial\Omega : |x_1| \leq 20, x_2 = 100\},$$
$$\Gamma_2 = \{(x_1,x_2) \in \partial\Omega : |x_1| > 20, x_2 = 100\}.$$

The material properties of the porous medium are given by Table 8. Fig. 12 shows the values of numerical pressure at the terminal time and there is no pressure oscillation. From Fig. 13, we observe that the arrows of numerical displacement near the boundaries match very well with ones on the boundaries, which fits the physical phenomenon very well.

Table 8: Values of parameters.

Parameters	Description	Values	Unit
λ^*	Coefficient of secondary consolidation	1e-2	-
ν	Poisson ratio	0.2	-
b_0	Biot-Willis constant	1	-
E	Young's modulus	3e4	N/m^2
β	Lamé constant	8.333e3	N/m^2
K	Permeability tensor	(1e-15) $\mathbf{I}$	m^2
γ	Lamé constant	1.25e4	N/m^2
a_0	Constrained specific storage coefficient	2e-8	-
θ_f	Fluid viscosity	1e-3	Pa s

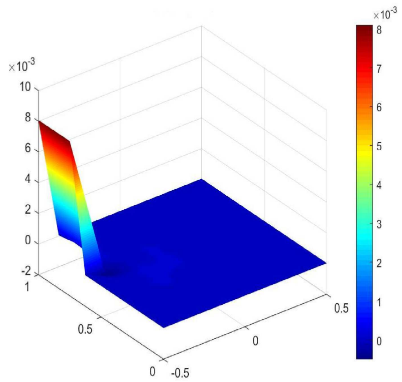

Figure 12: The numerical pressure p_h^{n+1} at the terminal time T with the parameters of Table 8 for the reformulated model.

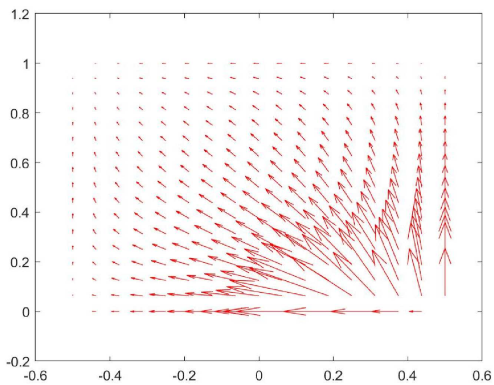

Figure 13: Arrow plot of the computed displacement $\boldsymbol{\tau}_h^{n+1}$ with the parameters of Table 8 for the reformulated model.

5 Conclusion

In this paper, we propose a new multiphysics finite element method for a Biot model with secondary consolidation in soil dynamics. To better describe the processes of deformation and diffusion underlying in the original model, we introduce new variables $q = \mathrm{div}\boldsymbol{\tau}, \varpi = a_0 p + b_0 q$ and $\delta = b_0 p - \beta q - \lambda^* q_t$ to reformulate the fluid-solid coupling problem to a fluid coupled problem, the multiphysics approach is different from the introduced variables in [11]. To the best of our knowledge, it is a completely new method to give the MFEA for the Biot model with secondary consolidation. The MFEA has an optimal convergence order and has a built-in mechanism to overcomes the "locking phenomenon" of pressure and displacement because the generalized Stokes problem (3.5)-(3.6) is not a saddle point problem for the finite constant β. Thus, the finite element spaces $(\mathbf{X}_h, M_h)$ can be chosen arbitrarily when the parameter β is finite, and one can choose stable Stokes solver when the parameter β is large enough or infinite. Also, we show some numerical examples to verify the theoretical results and the MFEA has no "locking phenomenon" of pressure and displacement.

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