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A NATURAL GRADIENT ALGORITHM FOR THE SOLUTION OF LYAPUNOV EQUATIONS BASED ON THE GEODESIC DISTANCE*

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Abstract

A new framework based on the curved Riemannian manifold is proposed to calculate the numerical solution of the Lyapunov matrix equation by using a natural gradient descent algorithm and taking the geodesic distance as the objective function. Moreover, a gradient descent algorithm based on the classical Euclidean distance is provided to compare with this natural gradient descent algorithm. Furthermore, the behaviors of two proposed algorithms and the conventional modified conjugate gradient algorithm are compared and demonstrated by two simulation examples. By comparison, it is shown that the convergence speed of the natural gradient descent algorithm is faster than both of the gradient descent algorithm and the conventional modified conjugate gradient algorithm in solving the Lyapunov equation.

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Key words: Lyapunov equation, Geodesic distance, Natural gradient descent algorithm.

1. Introduction

Lyapunov matrix equation is widely used in many fields, such as linear control systems, stochastic analysis of dynamical systems and circuit simulations, see, e.g., [1–5]. The solution of this equation has been paid more and more attention in the computational mathematics field, see, e.g., [6–9]. In particular, Li and White presented a Cholesky factor-alternating direction implicit algorithm to generate a low-rank approximation solution of the Lyapunov equation [10]. Vandereycken and Vandewalle provided a Riemannian optimization approach to compute the low-rank solution of the Lyapunov equation [11]. Jbilou proposed a preconditioned Krylov method for solving Lyapunov matrix equations [12]. Deng, Bai and Gao designed iterative orthogonal direction methods according to the fundamental idea of the classical conjugate direction method for the standard system of linear equations to obtain the Hermitian solutions of the linear matrix equations $AXB = C$ and $(AX, XB) = (C, D)$ [13]. Recently, Su and Chen

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proposed a modified conjugate gradient algorithm (MCGA) to solve Lyapunov matrix equations and some other linear matrix equations, which seemed to be the generalized result of [14].

Up to date, however, there has been few investigation on the solution problem of the Lyapunov matrix equation in the view of Riemannian manifolds. Note that this solution is a positive definite symmetric matrix in a global asymptotically stable linear system and the set of all the positive definite symmetric matrices can be taken as a manifold. Thus, it is more convenient to investigate the solution problem with the help of these attractive features on the manifold. To address such a need, we focus on a numerical method to solve the Lyapunov matrix equation on the manifold.

In the present paper, with the help of a natural gradient descent algorithm (NGDA), a new framework is proposed to calculate the numerical solution of the Lyapunov matrix equation on the manifold of positive definite matrices. Moreover, the geodesic distance on the curved Riemannian manifold is taken as the objective function of the NGDA. In order to compare with this NGDA, a gradient descent algorithm (GDA) based on the classical Euclidean distance is provided. Furthermore, the behaviors of NGDA, GDA and the conventional MCGA are compared and demonstrated by two simulation examples.

The remainder of the paper is organized as follows. The geometric structures on the manifold of positive definite matrices are introduced in Section 2. Then the GDA and NGDA are proposed to solve the Lyapunov matrix equation in Sections 3. Finally, two numerical examples are provided to compare the convergence speed and the computation consumption of the presented algorithms and the MCGA.

2. Preliminaries

Let $\mathrm{M}(n)$ be the set of $n \times n$ real matrices and $\mathrm{GL}(n)$ be its subset containing only nonsingular matrices. Then $\mathrm{GL}(n)$ is a Lie group, namely, a group which is also a differentiable manifold and on which the operations of the group multiplication and the inverse are smooth. The tangent space at the identity of $\mathrm{GL}(n)$ is called the corresponding Lie algebra, which is a space which consists of all linear transformations, namely $\mathrm{M}(n)$. The set of all the $n \times n$ positive definite symmetric matrices can be taken as a manifold, denoted by $\mathrm{PD}(n)$. Manifold $\mathrm{PD}(n)$ is not a Lie subgroup of $\mathrm{GL}(n)$, while it is a submanifold of $\mathrm{GL}(n)$. On manifold $\mathrm{PD}(n)$, we can define different metrics so that different geometry structures are established on this manifold. In this section, we briefly introduce these geometric structures which will be used in the following sections. More details can be found in [15–18].

2.1. Tangent space on manifold $\mathrm{PD}(n)$

The exponential map of a matrix $U \in \mathrm{M}(n)$ is given, as usual, by the convergent series

$$\exp(U) = \sum_{m=0}^{\infty} \frac{U^m}{m!}. \tag{2.1}$$

The inverse map, i.e., the logarithmic map is defined as follows

$$\log(V) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(V-I)^m}{m}, \tag{2.2}$$

for V in a neighborhood of the identity I of $\mathrm{GL}(n)$.