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FOURTH-ORDER COMPACT SCHEMES FOR HELMHOLTZ EQUATIONS WITH PIECEWISE WAVE NUMBERS IN THE POLAR COORDINATES*

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Abstract

In this paper, fourth-order compact finite difference schemes are proposed for solving Helmholtz equation with piecewise wave numbers in polar coordinates with axis-symmetric and in some cases that the solution depends both of independent variables. The idea of the immersed interface method is applied to deal with the discontinuities in the wave number and certain derivatives of the solution. Numerical experiments are included to confirm the accuracy and efficiency of the proposed method.

Mathematics subject classification: 65M06, 65N06.
Key words: Helmholtz equation, Compact finite difference schemes, Polar coordinate, The immersed interface method, High order method.

1. Introduction

Helmholtz equations can be used to model some important physical phenomena, such as acoustic wave scattering, noise reduction in silencers, water wave propagation etc. Many efforts have been made to develop more efficient and accurate numerical methods for the solution of Helmholtz equation, such as the boundary element method [13], finite element methods [11,12] and a few finite difference methods mentioned below. In [1], Harari and Turkel developed schemes with fourth order accurate local truncation errors on uniform meshes and third order in the nonuniform case. The methods are based on Padé expansions, and was extended by Singer and Turkel [2] to Neumann boundary conditions. Another method of approximating the Helmholtz equation with higher order accuracy were developed in [3-5]. Applications of the Helmholtz equation with discontinuous media can be found, for example, in [6,7]. Those methods mentioned above have been successfully developed in solving the Helmholtz equation when k is constant. When k is a piecewise constant, it becomes more difficult to develop high order methods [14,15]. Their schemes can keep global higher-order accuracy in the presence of discontinuities with piecewise wave number.

In this paper, we consider high order finite difference method for the Helmholtz equations with a piecewise wave number in the polar coordinates. The wave number is assumed to be piecewise constant and has a finite jump across interfaces that are few isolated circles. We

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propose fourth order compact schemes for Helmholtz equation with discontinuous coefficient in polar coordinates for the axis-symmetric and some cases that the solution depends both of independent variables by exploiting the idea introduced in [10] and the immersed interface method [8,9].

The rest of paper is organized as follows. We construct the compact scheme for the axis-symmetric Helmholtz equation in the polar coordinates with discontinuous wave number and show some numerical results to confirm our conclusion in the next section. Then, we derive a compact fourth order scheme for some cases that the solution depends both of independent variables in the polar coordinates followed by numerical examples. We present our conclusions and discussions of possible future directions in this subject.

2. The Axis-symmetric Problem

We start our discussion for the axis-symmetric which becomes essentially a one dimensional problem in polar coordinates

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+k^2u=f, \qquad r\in D^-\cup D^+, \quad a\le r\le b, \tag{2.1}$$

where wave number k is a piecewise constant, α is an interface, which separates the region into two parts $D^-=(a,\alpha)$ and $D^+=(\alpha,b)$, The jump conditions are defined as the difference of the limiting values from two different sides of interface

$$[u]_{r=\alpha}=\lim_{r\to\alpha,r\in D^+}u\left(r\right)-\lim_{r\to\alpha,r\in D^-}u\left(r\right)=u^+\left(\alpha\right)-u^-\left(\alpha\right),$$

across the interface, the solution satisfies the following natural jump conditions

$$[u]=0, \quad [u_r]=0. \tag{2.2}$$

2.1. Fourth-order compact scheme at regular and irregular points

Without loss of generality, we generate a uniform mesh in the interval (a,b), $r_i=(i-1)h$, $i=1,...,M+1$, where $h=(b-a)/M$ is the mesh size. Exploiting the fourth-order compact finite difference scheme introduced in [10], our fourth-order compact finite scheme can be written as

$$b_0u_{m+1}+b_1u_m+b_2u_{m-1}=F_m, \tag{2.3a}$$

where

$$b_0=\frac{r_{m+\frac{1}{2}}}{h^2}+\frac{h}{12r_m^3}, \tag{2.3b}$$

$$b_1=\frac{r_{m+\frac{1}{2}}+r_{m-\frac{1}{2}}}{h^2}-k^2\left(-\frac{h^2}{12r_m^2}+\frac{h^2k^2}{12}-1\right), \tag{2.3c}$$

$$b_2=\frac{r_{m-\frac{1}{2}}}{h^2}-\frac{h}{12r_m^3}, \tag{2.3d}$$

$$F_m=\left(1-\frac{k^2h^2}{12}+\frac{h^2}{12r_m^2}\right)f_m+\frac{h^2}{12r_m}f_m^{'}+\frac{h^2}{12}f_m^{''}, \tag{2.3e}$$

and $f=f(r)$ is a known function. We can solve Eq. (2.1) by scheme (2.3) when k is a constant. However, we have to modify scheme (2.3) for keeping high order accuracy when k is a piecewise