Journal of Computational Mathematics
Vol.34, No.5, 2016, 462–478.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1603-m2014-0193

A COMPACT FOURTH-ORDER FINITE DIFFERENCE SCHEME FOR THE IMPROVED BOUSSINESQ EQUATION WITH DAMPING TERMS*

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Abstract

In this paper, a compact finite difference method is presented for solving the initial boundary value problems for the improved Boussinesq equation with damping terms. The fourth-order equation can be transformed into a first-order ordinary differential system, and then, the classical Padé approximation is used to discretize spatial derivative in the nonlinear partial differential equations. The resulting coefficient matrix for the semi-discrete scheme is tri-diagonal and can be solved efficiently. In order to maintain the same order of convergence, the classical fourth-order Runge-Kutta method is the preferred method for explicit time integration. Soliton-type solutions are used to evaluate the accuracy of the method, and various numerical experiments are designed to test the different effects of the damping terms.

Mathematics subject classification: 65N30.
Key words: Compact finite difference method, Improved Boussinesq equation, Stokes damping, Hydrodynamic damping, Runge-Kutta method.

1. Introduction

The Boussinesq equation was first presented by Joseph Boussinesq in 1873, which is a fourth-order nonlinear PDE belonging to the Kdv family. This equation mainly describes the propagation of long waves on the free surface of shallow water under gravity conditions, which has been widely used in math-physical field related to nonlinear wave phenomena, such as ion-sound in plasma and, nonlinear lattice waves [1, 2]. In general, the equation has the following form:

$$u_{tt}(x,t) = u_{xx}(x,t) + qu_{xxxx}(x,t) + (u^2(x,t))_{xx}\,,$$

where $q = \pm 1$. When $q = 1$, it is the so-called bad Boussinesq equation and is ill-posed; whereas when $q = -1$, it is called a good Boussinesq equation and is well-posed. Many well known methods, such as the inverse scattering transform method, bilinear formalism, and the Bäcklund transformation method, can be used to handle the completely integrable Boussinesq equation [3]. Bogolubsky [4, 5] showed that the bad Boussinesq equation describes a spurious

* Received January 26, 2015 / Revised version received December 14, 2015 / Accepted March 21, 2016 / Published online September 14, 2016 /

instability of short wavelengths and its Cauchy problem is incorrect. Consequently Makhankov [2] demonstrated that the bad Boussinesq equation could be approximated using the improved Boussinesq equation (IBq for short) by replacing the term u_{xxxx} with u_{xxtt}:

$$u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxtt} = 0. \tag{1.1}$$

The Eq. (1.1) arises in acoustic waves on elastic rods with circular cross-sections when transverse motion and nonlinearity are examined [6], moreover, the improved Boussinesq equation is used to describe wave propagation at right angles to the magnetic field and is more suitable for numerical simulation than the bad Boussinesq equation. To some extent, it has been explored theoretically by several authors. Abdou [7] obtained the generalized solution and period solution of (1.1) using the Exp-function method. The sine or cosine ansatz method was utilized by Wazwaz [8] to find many compact and non-compact solutions of (1.1) with variants. In spite of these special soliton-type solutions, it is difficult to find an analytical solution satisfying a particular choice of initial conditions in most situations. Therefore, the main approach depends on solving the problem numerically.

The finite difference method is popular owing to its simplicity and ease of manipulation. Bogolubsky [5], Iskandar and Jain [9] were the first to study (1.1) using a three-level implicit nonlinear scheme with second-order accuracy. Later, Zoheiry [10] provided a three-level iteration relation based on the implicit compact difference scheme to improve the accuracy in space. Bratsos [11] used the typical second-order difference method to reduce (1.1) to a system of ordinary differential equations and developed a predict-correction scheme based on two different proper Padé approximations for the same matrix index term. The stability analysis was also presented. The scheme is complicated and the computational effort required is considerable.

The finite element method is also widely used for its geometric flexibility, Dursun and Irk [12] proposed two difference schemes and two quintic B-spline finite element collocation methods based on the second and third-order time discretization methods, these schemes were compared with each other subsequently. Lin and Wu [13] presented a finite element scheme based on the linear B spline. Inc and Evans [14] solved (1.1) using the domian decomposition method, which is a series expansion method, and the solution is expressed as a convergent series. An approximation for the solution is obtained by truncating the series after retaining a sufficient number of terms. Indeed, it is more accurate than the finite element method but has a great computational load.

The finite volume element method has widespread use due to its capability for local conservation. Zhang and Lu [15] developed a quadratic finite volume element method for (1.1). Wang and Zhang [16] applied this method to the stochastic damped Improved Boussinesq equation. This method seems to yield better results than the ordinary central finite difference method, but has the same second-order accuracy.

In this paper, we consider the damped Improved Boussinesq equation of the following form:

$$u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxtt} = -\tau_s u_t + \tau_h u_{txx}, \tag{1.2}$$

where in the right-hand terms, $\tau_s u_t, \tau_h u_{txx}$ denote the effect of Stokes damping and hydrodynamic damping [17, 18] respectively, and the coefficients should satisfy $\tau_s \geq 0$ and $\tau_h \geq 0$. It is clear that the damped IBq (1.2) reduces to the IBq (1.1) when $\tau_s = \tau_h = 0$.

The compact finite difference scheme [21, 22] has enjoyed great popularity in computational dynamics and electromagnetic and computational acoustics due to its high accuracy, compact stencils, and high resolution compared with the ordinary central finite difference scheme. We