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AN EFFICIENT NUMERICAL METHOD FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH TWO CAPUTO DERIVATIVES*

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Abstract

In this paper, we study the Hermite cubic spline collocation method with two parameters for solving a initial value problem (IVP) of nonlinear fractional differential equations with two Caputo derivatives. The convergence and nonlinear stability of the method are established. Some illustrative examples are provided to verify our theoretical results. The numerical results also indicate that the convergence order is $\min\{4-\alpha, 4-\beta\}$, where $0<\beta<\alpha<1$ are two parameters associated with the fractional differential equations.

Mathematics subject classification: 65L20, 65L60, 34A08.
Key words: Fractional differential equations; Caputo derivatives; Spline collocation method; Convergence; Stability.

1. Introduction

In this paper, we study the numerical method for the following initial value problem

$$\begin{cases} {}_0^CD_t^\alpha y(t) = f(t, y(t), {}_0^CD_t^\beta y(t)), \quad t\in I=[0,T], \quad 0<\beta<\alpha\le 1, \\ y(0)=y_0, \end{cases} \tag{1.1}$$

where $y(t)$ is at least a first order differentiable function, $f:[0,T]\times R\times R\to R$ is a given continuous mapping which satisfies

$$|f(t,y,u)-f(t,z,v)| \le L_1|y-z| + L_2|u-v\,|, \tag{1.2}$$
$$|I^\alpha[f(t,y,u)-f(t,z,v)]| \le \mu_1|I^\alpha(y-z)| + \mu_2|I^\alpha(u-v)|, \tag{1.3}$$

for all $t\in I=[0,T]$, $y,z,u,v\in C[0,T]$, where L_1, L_2, μ_1, μ_2 are positive constants, ${}_a^CD_t^\alpha$ denotes the Caputo fractional derivative of order α as

$${}_a^CD_t^\alpha y(t) = \begin{cases} I^{1-\alpha}\frac{d}{dt}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{d}{d\tau}y(\tau)d\tau, \quad t\ge 0, \quad if\ 0<\alpha<1, \\ \frac{d}{dt}y(t), \quad if \quad \alpha=1, \end{cases} \tag{1.4}$$

and I^α denotes the integral operator of order α as

$$I^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau, \quad t\ge 0, \quad if \quad \alpha>0, \\ I, \quad if \quad \alpha=0, \end{cases}$$

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here I is the identity operator. It is well known that (1.1) is equivalent to the following problem

$$\begin{cases} y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau), \ {}_0^C D_\tau^\beta y(\tau)) d\tau, \\ y(0) = y_0. \end{cases} \tag{1.5}$$

In the past decades, fractional calculus has been a hot spot of research in science and engineering [1,7,14,17–19]. The numerical approach to fractional differential equations(FDEs) has been recently discussed by numerous researchers [3–6,9–14,21]. In the last few years, spline collocation methods have been also successfully applied to many IVPs of FDEs. In 2011, Li, Chen [8] have used high order piecewise interpolation polynomial to approximate fractional integrals and fractional derivatives. In [8], they have used Simpson method to design a high order algorithm for FDEs. Analyses of error and stability have also been given. Pedas, Tamme [15,16] have successfully obtained the numerical solutions of some classes of IVPs of linear multi-order FDEs (MFDEs) by using spline collocation method, and also studied the convergence of the spline collocation method. But they didn't use these methods to solve IVPs of nonlinear MFDEs directly.

It's noteworthy that most of the above reviewed methods were designed to solve IVPs of Volterra integral equations which are equivalent to IVPs of FDEs. So we can see that there are not enough efficient numerical approaches to some applications. In [20], the Hermite cubic spline collocation method with two parameters for IVPs of FDEs has been discussed. This method is designed to solve IVPs of FDEs directly. Some illustrative examples successfully verify the obtained theoretical results. In this paper, we extend the work in [20], and the Hermite cubic spline collocation method is designed to solve IVPs of nonlinear FDEs with two Caputo derivatives directly. And the corresponding theoretical results including the local truncation error analysis, the convergence and the nonlinear stability of Hermite cubic spline collocation method for IVPs of FDEs with two Caputo derivatives are given.

This paper is organized as follows. In Section 2, we propose the spline cubic collocation method for solving IVPs of nonlinear FDEs with two Caputo derivatives directly, and the corresponding theoretical results are also given. Finally, we verify our theoretical analysis and show numerically the convergence order is $\min\{4-\alpha, 4-\beta\}$ by some examples in section 3.

2. The Hermite Cubic Spline Collocation Method

2.1. Numerical methods

Let $t_i = ih, \ 0 \le i \le N, h = T/N$, with N being a positive integer, be the grid points of the uniform partition of $[0,T]$ into the subintervals $I_i = [t_{i-1}, t_i], \ i = 1, \cdots, N$. On each subinterval I_i, the Hermite cubic spline function $S(t,h)$ can be represented by

$$S(t,h) = S_i(t,h),$$

where

$$\begin{aligned} S_i(t,h) = &\left(1 + 2\frac{t - t_{i-1}}{h}\right)\left(\frac{t_i - t}{h}\right)^2 S_{i-1}^{(0)} + \frac{(t-t_{i-1})(t_i - t)^2}{h^3} S_{i-1}^{(1)} \\ &+ \left(1 + 2\frac{t_i - t}{h}\right)\left(\frac{t - t_{i-1}}{h}\right)^2 S_i^{(0)} - \frac{(t-t_{i-1})^2(t_i - t)}{h^3} S_i^{(1)}, \end{aligned} \tag{2.1}$$