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MAXIMUM NORM ESTIMATE, EXTRAPOLATION AND OPTIMAL POINTS OF STRESSES FOR THE FINITE ELEMENT METHODS ON THE STRONGLY REGULAR TRIANGULATION*

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Abstract

Under the condition that the triangulation of the given domain is strongly regular, the maximum norm estimate with accuracy $O(h^2)$ of the linear finite element approximation is obtained, the optimal points of stresses at the midpoints of common sides for all adjacent elements are shown, and the estimate with higher accuracy for the extrapolation approximation based on mesh refinement and extrapolation is given.

§1. Introduction

The L^∞-error estimates of the finite element approximations for second order linear elliptic boundary value problems have been established by Frehse, Nitsche, Rannacher, Scott, et al. Fried has published an example which indicates that the pointwise estimate

$$\|u-u^h\|_{0,\infty}\leqslant ch^2\ln\frac{1}{h}\|u\|_{2,\infty} \tag{1}$$

may be of optimal order in the usual case. However, if some restrictive assumptions are imposed, then the convergence order can be improved. As an example, when the triangulation of the given domain is strongly regular (see [5]—[7] or the next section), and $u\in H^3(\Omega)\cap W^2_\infty(\Omega)$, the following result is obtained for the linear finite element approximation in [7]:

$$\|u-u^h\|_{0,\infty,D}\leqslant ch^2\left(\ln\frac{1}{h}\right)^{1/2}[\|u\|_{2,\infty,D}+\|u\|_{3,\Omega}], \tag{2}$$

where $D\subseteq\Omega$.

In the present paper, we shall prove in section 2 the following

Theorem 1. *If the triangulation Π_h of the given domain Ω is strongly regular and $u\in W^3_\infty(\Omega)\cap H^1_0(\Omega)$, then the pointwise accuracy of the linear finite element approximation u^h will be*

$$\max_{p\in\Omega}|u(p)-u^h(p)|\leqslant ch^2\|u\|_{3,\infty}. \tag{3}$$

One of the new developments in finite element analysis is the investigation of the phenomena of superconvergence. Obviously, it is of interest to improve the accuracy of stresses by using the optimal points of stresses. The superconvergence

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estimate for the gradient with accuracy $O(h^2)$ has been obtained in [5, 6] by using the means of gradients for two adjacent elements as the approximations to the gradient at the midpoints of common sides for some elements. However, it was not proved that there exist the optimal points of stresses for all elements and it was not stated where the elements which have the optimal points of stresses are located. In [8], the above result was improved and the inner superconvergence estimates of gradient for the optimal point of stresses was obtained. In section 3, we shall prove the following.

Theorem 2. *If the triangulation Π_h is strongly regular and $u \in C^3(\Omega) \cap H_0^1(\Omega)$, then the midpoints of common sides for all adjacent elements are the entirely optimal points of stresses with accuracy $O\left(h^2 \ln \frac{1}{h}\right)$.*

In the last section, the extrapolation for the finite element approximations is considered. The mesh refinement for the triangulation Π_h of Ω should be a new triangulation which is achieved via dividing each triangle of Π_h into four small equal triangles. Let u^h be the linear finite element approximation over the triangulation Π_h and $u^{h/2}$ the new approximation over a new triangulation. The numerical results show that the accuracy of the extrapolation approximation $\frac{1}{3}(4u^{h/2}-u^h)$ is much better than $u^{h/2}$. However, the theoretical basis for this algorithm still remains an open question, and we will try to give some answer to this question. We prove.

Theorem 3. *Assume that the triangulation Π_h is strongly regular and $u \in C^4(\Omega) \cap H_0^1(\Omega)$. Let u^h, $u^{h/2}$ be the linear finite element approximation over Π_h and the refinement respectively. We have for the nodes of Π_h*

$$u-\frac{1}{3}(4u^{h/2}-u^h)=O\left(h^3 \ln \frac{1}{h}\right). \tag{4}$$

§ 2. Maximum Norm Estimate

For simplicity we shall consider the 2-dimensional Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5}$$

Suppose that the triangulation Π_h of $\Omega_h (\subseteq \Omega)$ is strongly regular, i. e. Π_h satisfies the following conditions:

c1: Each triangle $T \in \Pi_h$ contains a circle of radius $c_1 h$ and is contained in a circle of radius $c_2 h$, $0<c_1<c_2$ independent of h and T (quasi-uniform).

c2: Any two adjacent triangles of Π_h form an approximate parallelogram, i. e. there exists a constant c independent of h, such that (see Fig. 1)

$$|\overline{p_1p_2}-\overline{p_3p_4}| \leqslant ch^2. \tag{6}$$

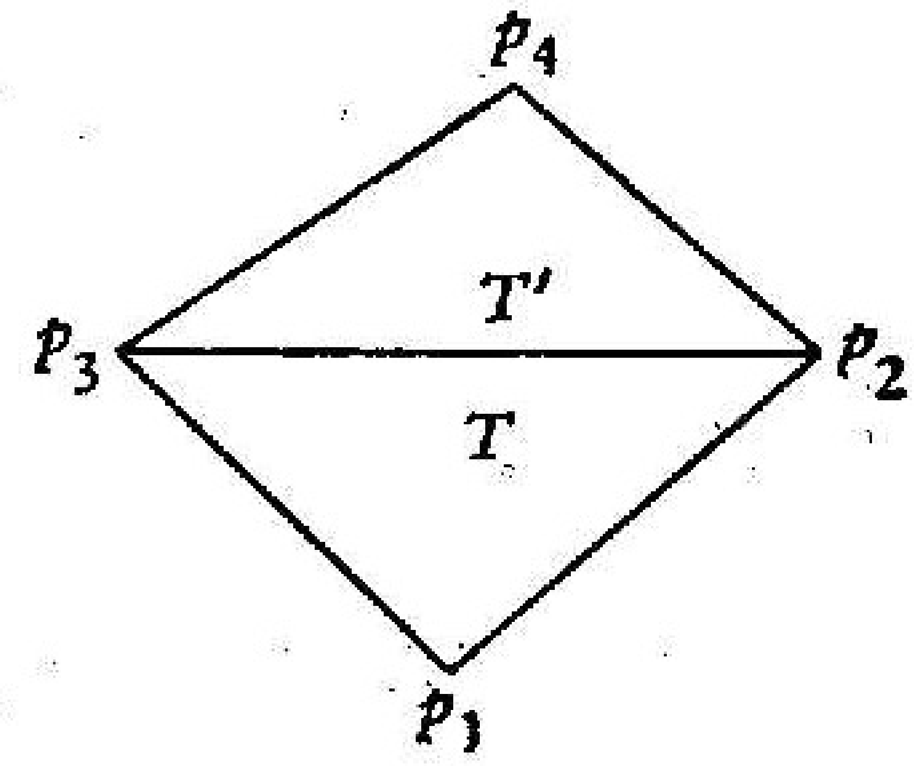

Fig. 1

Let S_h be the piecewise linear finite element space on Ω_h with zero on $\Omega \backslash \Omega_h$, $u^h \in S_h$ the finite element approximation satisfying